

From elliptic fixed points of 2-d diffeomorphisms  
to dynamics in the circle and the annulus  
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**Abstract**

Perturbing the germ at the origin of a planar rotation  $re^{2\pi i\theta} \mapsto re^{2\pi i(\theta+\omega)}$  leads to two celebrated results which describe geometrically the *dynamical* behaviour of the iterates of the perturbed diffeomorphism  $F$ , that is the structure of the *orbits*  $\mathcal{O}(z) = \{z, F(z), F^2(z), \dots, F^n(z), \dots\}$ : the *Andronov–Hopf–Neimark–Sacker bifurcation* of invariant curves under a generic radial hypothesis of weak attraction (or repulsion) and the *Moser invariant curve* theorem under an angular twist hypothesis in the area preserving case. The invariant curves whose existence is proved are normally hyperbolic with generic induced dynamics in the first case, with a dynamics smoothly conjugate to a diophantine rotation in the second one.

Statements and proofs in this first section illustrate the notion of *normal form*, introduced by Poincaré in his thesis in 1879. Closely related to the “averaging of perturbations” used by astronomers since the eighteenth century, it generalizes the Jordan normal form of a matrix to the non-linear world. Namely, by introducing local coordinates which reveal an approximate geometry underlying the situation, it sets the scene for the application of refined analytic tools to the determination of which features of this geometry do really exist.

The two sections which follow are devoted to the dynamical study of the objects which appeared in the first: homeomorphisms of the circle and monotone distortions of the annulus. In the last section, a hint is given of the intermediate dynamics of non invertible endomorphisms of the circle.

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# 1 Elliptic fixed points

Let  $F : (S, p) \rightarrow (S, p)$  be a local  $C^\infty$  (or analytic) diffeomorphism of a surface  $S$  defined in the neighborhood of a fixed point  $p = F(p)$ . The fixed point is said to be elliptic if the spectrum of the derivative  $dF(p)$  is of the form  $\{2\pi i\omega, -2\pi i\omega\}$  with  $\omega \neq \pm 1$ . This is equivalent to the existence of a linear conjugation of  $dF(p)$  with the rotation of angle  $2\pi\omega$ . Hence, after choosing good coordinates, one can suppose that  $p = 0$  and that  $F : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$  is such that

$$F(\zeta) = \lambda\zeta + O(|\zeta|^2), \quad \text{with } \lambda = e^{2\pi i\omega}.$$

In other words,  $F$  is a perturbation of a rotation.<sup>1</sup> Now, a rotation preserves each circle centered at the origin. This is a very strong property, very likely to be destroyed by the non-linear terms in the Taylor expansion of  $F$ . Nevertheless, reality is subtler and the study of the fate of these invariant circles is the starting point of two famous theories which correspond roughly to the dichotomy between *dissipative* and *conservative* dynamics:

1) *Andronov–Hopf–Neimark–Sacker bifurcation theory* which analyzes what happens when one considers a *generic*<sup>2</sup> diffeomorphism  $F$  with an elliptic fixed point at 0. The local behaviour of  $F$  itself is quite dull: indeed, the radial behaviour of the nonlinear terms turns the fixed point into an attractor or a repulsor and no other invariant object persists in its neighborhood. It is only when considering “generic” 1-parameter families  $F_\mu$  of local diffeomorphisms stemming from  $F_0 = F$  that the whole richness of the dynamics is regained (see [A1, A2]): each small enough circle invariant under the rotation  $dF(0)$  becomes a *normally hyperbolic*<sup>3</sup> closed curve invariant under some  $F_\mu$  (figure 3).

2) *Kolmogorov–Arnold–Moser (KAM) theory* which analyzes the case when  $F$  is area preserving, a hypothesis which is natural for diffeomorphisms with a mechanical origin, the paradigmatic example being *first return maps*<sup>4</sup> in the *restricted three body problem* first studied by Poincaré (see [C2], section 6). In this case, it is the angular behaviour of the non-linear terms which plays the key part, the result being that “many” of the circles invariant under the rotation  $dF(0)$  persist in the form of closed curves invariant under the action of  $F$  itself. Moreover the restriction of  $F$  to such an invariant closed curve is smoothly conjugated to a rotation whose angle is of the form  $2\pi\alpha$  with  $\alpha$  not rational and even “far from the rationals” in a precise sense.

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<sup>1</sup>Beware that the notation  $F(\zeta)$  does not mean that  $F$  is complex analytic, its expression depends on  $\zeta$  and  $\bar{\zeta}$

<sup>2</sup>We shall not give a formal definition of this word; it means essentially that what is described is the general situation and that only special hypotheses could prevent the description to be correct.

<sup>3</sup>Roughly speaking this means that any attraction or repulsion normal to the curve under the iterates of  $F_\mu$  dominates any attraction or repulsion inside the curve; this condition insures the robustness of the curve

<sup>4</sup>see section 1.4 of [C1] for a brief introduction

## 1.1 Preparation: Poincaré's theory of normal forms

The idea, which goes back to Poincaré's thesis in 1879, is the following: being a rotation, the derivative of  $F$  commutes with the whole group  $SO(2)$  of rotations. This is shown to imply that, provided some conditions on  $\omega$  are satisfied, a high order approximation of  $F$  is locally invariant by an action of  $SO(2)$  close to the standard one. Equivalently, one proves the existence of local coordinates which reveal the approximate geometry of the map, in a spirit similar to the Jordan form of a matrix. *The reader will notice that the existence of only a finite number of derivatives of  $F$  is required for what follows; this is important in view of a first reduction to a center manifold which, in general, has only finite differentiability.*

**Theorem 1** *If  $\lambda = e^{2\pi i\omega}$  is such that  $\lambda^q \neq 1$  for all integers  $q \in \mathbb{N}$  such that  $q \leq 2n + 2$ , there exists a local diffeomorphism*

$$H : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0), \quad \zeta \mapsto z = H(\zeta) = \zeta + O(|\zeta|^2)$$

such that

$$H \circ F \circ H^{-1}(z) = N(z) + O(|z|^{2n+2}), \quad \text{where } N(z) = z(1 + f(|z|^2))e^{2\pi i(\omega + g(|z|^2))},$$

with  $f$  and  $g$  real polynomials of degree  $n$  such that  $f(0) = g(0) = 0$ . If moreover  $\lambda^{2n+3} \neq 1$ , one can achieve a rest which is  $O(|z|^{2n+3})$ .

The so-called *normal form*  $N$ , is characterized by the fact that it commutes with the whole group  $SO(2)$  of rotations:

$$\forall \alpha, N(e^{2\pi i\alpha} z) = e^{2\pi i\alpha} N(z).$$

*Proof.* Let us start with a local diffeomorphism of degree 2,

$$H_2 : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0), \quad z = H_2(\zeta) = \zeta + \sum_{i+j=2} \gamma_{ij} \zeta^i \bar{\zeta}^j.$$

The direct computation of  $H_2 \circ F \circ H_2^{-1}$  is illustrated on the diagram below:

$$\begin{array}{ccc} \zeta & \xrightarrow{F} & \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \bar{\zeta}^j + O(|\zeta|^3) \\ \downarrow H_2 & & \downarrow H_2 \\ \zeta + \sum_{i+j=2} \gamma_{ij} \zeta^i \bar{\zeta}^j & \xrightarrow{H \circ F \circ H^{-1}} & \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \bar{\zeta}^j + \sum_{i+j=2} \gamma_{ij} \lambda^i \bar{\lambda}^j \zeta^i \bar{\zeta}^j + O(|\zeta|^3) \\ = z & & = \lambda z + \sum_{i+j=2} (\alpha_{ij} + \gamma_{ij}(\lambda^i \bar{\lambda}^j - \lambda)) z^i \bar{z}^j + O(|z|^3) \end{array}$$

Figure 1. Changing coordinates.



Supposing that  $F(\zeta) = \lambda\zeta + \sum_{i+j=2} \alpha_{ij} \zeta^i \bar{\zeta}^j + O(|\zeta|^3)$ , we get

$$H_2 \circ F \circ H_2^{-1}(z) = \lambda z + \sum_{i+j=2} \left( \alpha_{ij} + (\lambda^i \bar{\lambda}^j - \lambda) \gamma_{ij} \right) z^i \bar{z}^j + O(|z|^3).$$

Hence, if no *resonance* relation of the form  $\lambda^i \bar{\lambda}^j - \lambda = 0$  is satisfied with indices  $i, j$  such that  $i + j = 2$ , that is if  $\lambda^3 \neq 1$  (otherwise  $\bar{\lambda}^2 - \lambda = 0$ ), the choice of  $\gamma_{ij} = -(\lambda^i \bar{\lambda}^j - \lambda)^{-1} \alpha_{ij}$  kills all degree 2 terms in the Taylor expansion of the transformed map  $H_2 \circ F \circ H_2^{-1}$ .

If one tries in the same way to simplify the terms of degree 3 in the Taylor expansion of  $H_2 \circ F \circ H_2^{-1}$ , one stumbles upon an *unavoidable resonance*

$$\lambda^2 \bar{\lambda} - \lambda = 0$$

which merely reflects that  $|\lambda| = 1$ . Hence, if no other resonance of order 3 exists, which amounts to saying that  $\lambda^4 \neq 1$  (otherwise  $\bar{\lambda}^3 - \lambda = 0$ ), a local diffeomorphism  $H_3$  of the form  $H_3(z) = z + \sum_{i+j=3} \gamma_{ij} z^i \bar{z}^j$  can be found such that<sup>5</sup>

$$H_3 \circ H_2 \circ F \circ H_2^{-1} \circ H_3^{-1}(z) = \lambda z + c_1 z |z|^2 + O(|z|^4).$$

Now, if  $\lambda^q \neq 1$  for all  $q \leq 2n + 3$ , one finds by induction a local diffeomorphism  $H = H_{2n+2} \circ H_{2n+1} \circ H_3 \circ H_2$  tangent to  $Id$  at 0 such that

$$H \circ F \circ H^{-1}(z) = \lambda z + \sum_{k=1}^n c_k z |z|^{2k} + O(|z|^{2n+3}).$$

If  $\lambda^{2n+3} = 1$ , there is possibly a monomial  $\gamma \bar{z}^{2n+2}$  which cannot be canceled. Finally, choosing polar coordinates, one writes  $H \circ F \circ H^{-1}$  as in the conclusion of the theorem.

**Remark.** Resonances of the form  $\lambda^q = 1$  for  $1 \leq q \leq 4$  are called *strong resonances*. They are characterized by the fact that the resonant monomial  $\bar{z}^{q-1}$  is of smaller or comparable order to the first unavoidable resonant monomial  $z|z|^2$  and hence could play a role in the geometry of the normal form  $N$  which could become invariant only by rotations by an angle multiple of  $2\pi/q$ . *In the sequel, the hypotheses always exclude strong resonances.*

**Exercise 1** Write down the general form of a normal form  $N$  in case  $\lambda^q = 1$  and  $q \in \mathbb{N}$  is the smallest integer with this property. Show that, in general,  $N$  will only commute with the finite group generated by the rotation  $\frac{2\pi}{q}$ :

$$N(e^{\frac{2\pi}{q}i} z) = e^{\frac{2\pi}{q}i} N(z).$$

**Remark on notations.** : *Theorem 1* allows us to suppose from the start that local coordinates  $z$  have been chosen so that  $F$  is in the form given, by *Theorem 1*. In other words, from now on we shall write  $F(z)$  instead of  $H \circ F \circ H^{-1}(z)$ .

<sup>5</sup>in order to avoid too cumbersome notations we still call  $z$  the transformed coordinate  $H_3(z)$ .

## 1.2 The dissipative case

### 1.2.1 Andronov–Hopf–Neimark–Sacker bifurcation

The first two names are attached to the “continuous” case of a differential equation, the last two to the present “discrete” case of a map (see [A1, A2, I, C3]).

In general, the polynomial  $f(s) = \sum_{k=1}^n a_k s^k$  is such that  $a_1 \neq 0$ . If  $a_1 < 0$ , one can scale the coordinates so that  $a_1 = -1$  which, provided  $\lambda^q \neq 1$  for all integers  $1 \leq q \leq 4$ , puts  $F$  into the form

$$F(z) = N(z) + O(|z|^4), \quad \text{where } N(z) = z(1 - |z|^2)e^{2\pi i(\omega + b_1|z|^2)}.$$

As well as the rotation  $dF(0)$ , the normal form  $N$  still leaves invariant the foliation by circles centered at 0 but it sends the circle of radius  $r$  onto the circle of radius  $r(1 - r^2)$ . This implies not only that  $\lim_{m \rightarrow \infty} N^m(z) = 0$  but also that  $\lim_{m \rightarrow \infty} F^m(z) = 0$  as soon as  $|z|$  is small enough. Indeed, if  $|z|$  is small enough,  $|F(z)| < |z| |1 - \frac{1}{2}|z|^2|$ .

One says that 0 is a *weak attractor* (figure 2), the adjective “weak” recalling that the attraction is due to a non-linear term.

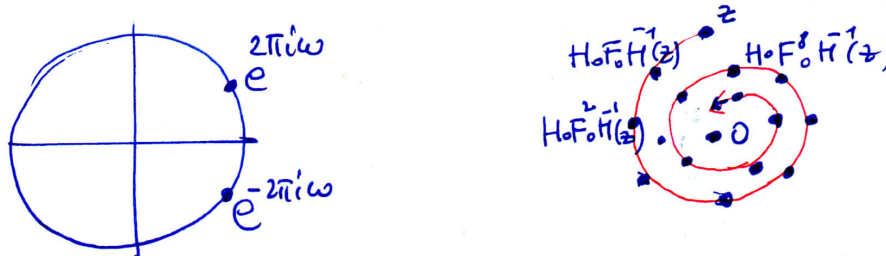


Figure 2. Weak attraction.

Hence we completely understand the dynamics of  $F$  in some neighborhood  $\mathcal{V}$  of the fixed point 0. Things become much more interesting if one perturbs  $F$  by including it in a smooth one parameter family of local diffeomorphisms  $F_\mu$  such that  $F_0 = F$ . As  $\frac{\partial F_0}{\partial z}(0) - Id$  is invertible, a direct application of the implicit function theorem shows that, in the neighborhood of 0, the equation  $F_\mu(z) - z = 0$  has a unique solution  $z_\mu$  depending smoothly on  $\mu$  and such that  $z_0 = 0$ . Hence, after a translation by  $z_\mu$  of the coordinates, one can suppose that for all  $\mu$  near 0, one has  $F_\mu(0) = 0$ .

For values of  $\mu$  such that the spectrum of  $dF_\mu(0)$  is not on the unit circle, there is no resonance and one could get a normal form which is linear up to any order. However, this would not be of much use: on the one hand the domain of definition of the conjugating diffeomorphism  $H_\mu$  tends to 0 when the spectrum of  $dF_\mu(0)$  tends to the unit circle and interesting phenomena occur outside of this domain, on the other hand, this would break the continuity with respect to  $\mu$  of the coordinate change  $H_\mu$ . In consequence, one chooses to eliminate in  $F_\mu$  only the same terms as the ones we have eliminated in  $F_0$ , that is we mimic for

$H_\mu$  the construction of  $H$  in section 1.1. Doing so one gets a smooth family  $H_\mu$  of local diffeomorphisms of  $(\mathbb{C}, 0)$  defined in a fixed neighborhood of 0 which put  $F_\mu$  into the form  $F_\mu(z) = z(1 + f_\mu(|z|^2))e^{2\pi i(\omega + g_\mu(|z|^2))} + \dots$  given by Theorem 1 except that  $f_\mu(s) = \sum_{i=0}^n a_i s^i$  and  $g_\mu(s) = \sum_{k=0}^n b_k(s) s^k$  now start with terms of degree 0. Finally, we shall suppose that  $a_0(\mu)$  is monotone (say increasing) for  $\mu$  close enough to zero. This is also a “generic” condition which amounts to saying that the spectrum of the derivative  $dF_\mu(0)$  crosses transversally the unit circle when  $\mu$  crosses the value 0. It allows us to change parameters and suppose that  $a_0(\mu) = \mu$ . At the end, we are reduced to study a family  $F_\mu$  of local diffeomorphisms of the form

$$\begin{cases} F_\mu(z) = N_\mu(z) + O(|z|^4), \text{ where} \\ N_\mu(z) = z(1 + \mu + a_1(\mu)|z|^2) e^{2\pi i(b_0(\mu) + b_1(\mu)|z|^2)}, \text{ and} \\ a_1(\mu) = -1 + O(|\mu|), \quad b_0(\mu) = \omega + O(|\mu|). \\ \text{The rest can be made } O(|z|^5) \text{ except if } \lambda^5 = 1, \text{ which can leave a term } \gamma \bar{z}^4. \end{cases}$$

Due to the commutation of  $N_\mu$  with the group  $SO(2)$  of rotations, the study of its dynamics reduces to an elementary question in dimension 1, namely the dynamics of the map from  $\mathbb{R}_+$  to itself  $r \mapsto r(1 + \mu + a_1(\mu)r^2)$ . The results are summarized in figure 3: the origin, which is a strong (=linear) attractor when  $\mu < 0$ , becomes a strong repeller when  $\mu > 0$ . But points far enough from the origin are still attracted and in between appears an invariant circle  $C_\mu$  of radius the unique solution  $r_\mu$  of the equation  $\mu + a_1(\mu)r_\mu^2 = 0$ . This circle is a *global attractor*: under iteration of  $N_\mu$ , it attracts any point of  $\mathbb{R}^2$  except the fixed point  $\{0\}$ . Moreover, it is *normally hyperbolic*, which means that every attraction or repulsion normally to it is stronger than any attraction or repulsion tangential to it, that is any attraction or repulsion of the rotation  $\theta \mapsto \theta + b_0(\mu) + b_1(\mu)r_\mu^2$ .

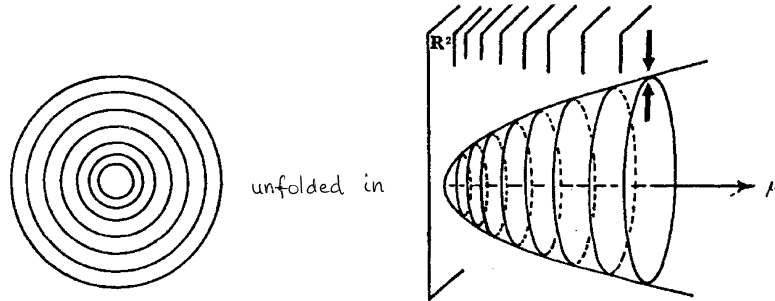


Figure 3. Dynamics of the family of normal forms  $N_\mu$ .

The content of the following theorem is that the perturbation from  $N_\mu$  to  $F_\mu$  is small enough so as not being able to destroy such a normally hyperbolic invariant curve (for a general theorem of persistence of normally hyperbolic invariant sets under a small enough perturbation, see [HPS]).

**Theorem 2 (Neimark 1959, Sacker 1964)** *Under the above hypotheses (in fact  $F_\mu$   $C^5$  is enough), for each  $\mu > 0$  small enough,  $F_\mu$  possesses a Lipschitz invariant closed curve  $\Gamma_\mu$ , close to  $C_\mu$ , which attracts a uniform (that is independent of  $\mu$ ) neighborhood  $\mathcal{V}$  of 0 (with 0 deleted). If the local diffeomorphisms  $F_\mu$  are of class  $C^\infty$ , these curves are of class  $C^k$  with  $k$  going to infinity when  $\mu$  tends to 0.*

*Proof.* We shall treat only the case when  $\lambda^5 \neq 1$ . If  $\lambda^5 = 1$  and the term  $\gamma\bar{z}^4$  is present, a circle is not a good enough approximation of the invariant curve and a further change of variables is necessary to get to a tractable form, see [1].

Setting  $z = re^{2\pi i\theta}$  and  $Z = F_\mu(z) = Re^{2\pi i\Theta}$ , we have

$$R = (1 + \mu)r + a_1(\mu)r^3 + O(r^5), \quad \Theta = \theta + b_0(\mu) + b_1(\mu)r^2 + O(r^4).$$

Now, the proof proceeds in two steps:

1) One encloses the invariant circle  $C_\mu$  in an annulus  $A_\mu$  of width  $O(|\mu|)$ , say the one bounded by the circles whose radii  $r_\mu^\pm$  are the two solutions of the equation  $\mu + a_1(\mu)r^2 \pm r^3 = 0$ . One checks that every point  $z \neq 0$  in some uniform (i.e. independent of  $\mu$ ) neighborhood  $\mathcal{V}$  of 0 is eventually sent inside  $A_\mu$  under the iterates of  $F_\mu$ .

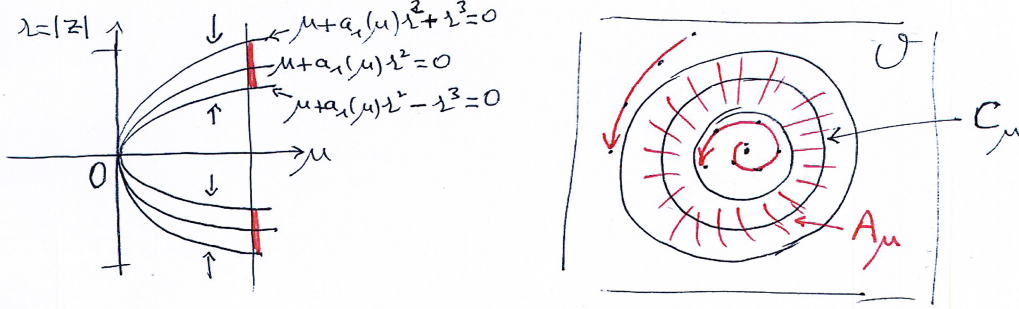


Figure 4. The attracting annulus  $A_\mu$ .

Indeed, such a uniform neighborhood may be defined by the two conditions:

$$\begin{cases} \text{if } \mu + a_1(\mu)r^2 + r^3 < 0, & R - r = \mu r + a_1(\mu)r^3 + O(r^5) < -\frac{1}{2}r^4, \\ \text{if } \mu + a_1(\mu)r^2 - r^3 > 0, & R - r = \mu r + a_1(\mu)r^3 + O(r^5) > \frac{1}{2}r^4. \end{cases}$$

2) One shows that under the iterates of  $F_\mu$ , every point inside the annulus tends asymptotically to some invariant curve  $\Gamma_\mu$  close to the circle  $C_\mu$ . For this, we choose coordinates in an annulus containing  $A_\mu$ , centered on  $C_\mu$  and of the form:

$$z = r_\mu(1 + \sqrt{\mu}\sigma) e^{2\pi i\theta},$$

where  $(\theta, \sigma) \in \mathbb{T} \times I$ ,  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ ,  $I = [-1, 1]$  (in these coordinates,  $A_\mu$  corresponds to  $\mathbb{T} \times [-1/2 + O(\sqrt{\mu}), 1/2 + O(\sqrt{\mu})]$ ). The map  $F_\mu$  becomes (we keep the same notation  $F_\mu$  for convenience)

$$F_\mu(\sigma, \theta) = \left( (1 - 2\mu)\sigma + \mu^{3/2}H(\sigma, \theta, \mu), \theta + \omega(\mu) + \mu^{3/2}K(\sigma, \theta, \mu) \right),$$

where  $\omega(\mu) = b_0(\mu) + b_1(\mu)r_\mu^2 = b_0(\mu) - b_1(\mu)\mu/a_1(\mu)$  and  $H(\sigma, \theta, \mu)$  and  $K(\sigma, \theta, \mu)$  are  $C^1$  with respect to all variables as soon as the original family  $F_\mu$  is at least  $C^5$  in  $z$  and  $C^1$  in  $\mu$ . The formula makes clear that, for  $\mu$  small enough, the normal contraction (in  $O(\mu)$ ) dominates the perturbation (in  $O(\mu^{3/2})$ ).

Let  $\{(\theta, \psi(\theta)) \subset \mathbb{T} \times I$  be the graph of a function  $\theta \mapsto \sigma = \psi(\theta)$  from the circle  $\mathbb{T}$  to  $I$ . We shall see that, as soon as  $\mu$  is small enough, the image by  $F_\mu$  of the graph  $\Gamma_\psi$  of  $\psi$ , is contained in  $\mathbb{T} \times I$  and is the graph of a function  $\mathcal{F}_\mu\psi : \mathbb{T} \rightarrow I$ :

$$F_\mu(\Gamma_\psi) = \Gamma_{\mathcal{F}_\mu\psi}.$$

The map  $\psi \mapsto \mathcal{F}_\mu\psi$  is called the *graph transform*. Thanks to the contracting factor  $1 - 2\mu$  which dominates any contraction along the angular direction (a manifestation of the fact that the *normal hyperbolicity* of  $C_\mu$  dominates the perturbation), one shall show that  $\mathcal{F}_\mu$  is a contraction in a well chosen Banach space of  $C^k$  functions provided  $\mu$  is close enough to 0 (a condition more and more stringent when  $k$  tends to  $+\infty$ ). The attracting invariant curve  $\Gamma_\mu \subset A_\mu$  we are looking for is the graph of the unique fixed point of this contraction.

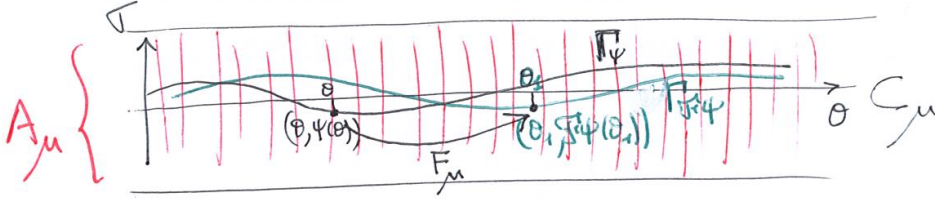


Figure 5. Graph transform.

More precisely, let us take as our space

$$A_{1,1}^0 = \{\psi : \mathbb{T} \rightarrow I, |\psi(\theta_1) - \psi(\theta_2)| \leq |\theta_1 - \theta_2|\}$$

endowed with the  $C^0$  norm  $\|\psi\|_0 = \sup_{\theta \in \mathbb{R}/\mathbb{Z}} |\psi(\theta)|$ .

**Exercise 2** *It is complete.*

**Lemma 3** *Let  $\psi \in A_{1,1}^0$ . If  $2\mu^{3/2}l < 1$ , the map  $g : \mathbb{T} \rightarrow \mathbb{T}$ , defined by*

$$g(\theta) = \theta + \omega(\mu) + \mu^{3/2}K(\psi(\theta), \theta, \mu),$$

*where  $l$  a Lipschitz constant of  $K$ , is a Lipeomorphism of the circle  $\mathbb{T}$ .*

*Proof.* Writing  $h(\theta) = \mu^{3/2}K(\psi(\theta), \theta, \mu)$ , we have for any  $\theta_1, \theta_2 \in \mathbb{T}$ ,

$$|h(\theta_1) - h(\theta_2)| \leq \mu^{3/2}l(|\psi(\theta_1) - \psi(\theta_2)| + |\theta_1 - \theta_2|) \leq 2\mu^{3/2}l|\theta_1 - \theta_2|.$$

Hence  $g$  is Lipschitz. If  $2\mu^{3/2}l < 1$ , its inverse  $g^{-1}$  is also Lipschitz; indeed, setting  $\hat{\theta} = g(\theta)$ , one checks immediately that  $|\theta_1 - \theta_2| \leq (1 - 2\mu^{3/2}l)^{-1}|\hat{\theta}_1 - \hat{\theta}_2|$ .

In order to end the proof of theorem 2 (except for the part which concerns the regularity of the invariant curve), it remains to show that the map  $\mathcal{F}_\mu$ , defined by

$$\mathcal{F}_\mu\psi(\theta) = (1 - 2\mu)\psi(g^{-1}(\theta)) + \mu^{3/2}H(\psi(g^{-1}(\theta)), g^{-1}(\theta), \mu),$$

possesses the following properties:

- 1) it sends the space  $A_{1,1}^0$  to itself;
- 2) it is a contraction for the  $C^0$ -norm on  $A_{1,1}^0$ ;
- 3) the graph of the unique fixed point of this contraction is an invariant closed curve which attracts every point of the annulus  $A_\mu$  under  $F$ .

*Proof of 1).* On the one hand,

$$\|\mathcal{F}_\mu\psi\|_0 \leq (1 - 2\mu)\|\psi\|_0 + \mu^{3/2}\|H\|_0 \leq 1 - 2\mu + O(\mu^{3/2})$$

is less than 1 as soon as  $\mu$  is small enough, on the other hand, denoting now by  $l$  a Lipschitz constant common to  $H$  and  $K$ ,

$$|\mathcal{F}_\mu\psi(\theta_1) - \mathcal{F}_\mu\psi(\theta_2)| \leq \frac{1 - 2\mu + 2\mu^{3/2}l}{1 - 2\mu^{3/2}l}|\theta_1 - \theta_2| < |\theta_1 - \theta_2|.$$

*Proof of 2)* With obvious notations,

$$\begin{aligned} |\mathcal{F}_\mu\psi_1(\theta) - \mathcal{F}_\mu\psi_2(\theta)| &\leq (1 - 2\mu)|\psi_1(g_1^{-1}(\theta)) - \psi_2(g_2^{-1}(\theta))| \\ &\quad + \mu^{3/2}|H(\psi_1(g_1^{-1}(\theta)), g_1^{-1}(\theta), \mu) - H(\psi_2(g_2^{-1}(\theta)), g_2^{-1}(\theta), \mu)| \\ &\leq (1 - 2\mu)(|g_1^{-1}(\theta) - g_2^{-1}(\theta)| + \|\psi_1 - \psi_2\|_0) \\ &\quad + \mu^{3/2}l(2|g_1^{-1}(\theta) - g_2^{-1}(\theta)| + \|\psi_1 - \psi_2\|_0) \\ &\leq (1 - 2\mu + \mu^{3/2}l)\|\psi_1 - \psi_2\|_0 + (1 - 2\mu + 2\mu^{3/2}l)|g_1^{-1}(\theta) - g_2^{-1}(\theta)| \\ &\leq \left(1 - 2\mu + \mu^{3/2}l + \frac{(1 - 2\mu + 2\mu^{3/2}l)\mu^{3/2}l}{1 - 2\mu^{3/2}l}\right)\|\psi_1 - \psi_2\|_0. \end{aligned}$$

In the penultimate inequality, the second term comes from the identity

$$\theta = g^{-1}(\theta) + \omega_\mu + \mu^{3/2}K(\psi(g^{-1}(\theta)), g^{-1}(\theta), \mu),$$

which implies

$$\begin{aligned} |g_1^{-1}(\theta) - g_2^{-1}(\theta)| &\leq \mu^{3/2}l(|\psi_1(g_1^{-1}(\theta)) - \psi_2(g_2^{-1}(\theta))| + |g_1^{-1}(\theta) - g_2^{-1}(\theta)|) \\ &\leq 2\mu^{3/2}l|g_1^{-1}(\theta) - g_2^{-1}(\theta)| + \mu^{3/2}l\|\psi_1 - \psi_2\|_0. \end{aligned}$$

*Proof of 3)* Let  $\psi_\mu \in A_{1,1}^0$  be the unique fixed point of the contraction  $\mathcal{F}_\mu$ . As the annulus represented by  $\mathbb{T} \times I$  contains  $A_\mu$ , it is enough to show that, under iteration of  $F_\mu$ , every point of  $\mathbb{T} \times I$  is attracted by the graph of  $\psi_\mu$ . But this follows immediately from the fact that under iteration of  $\mathcal{F}_\mu$  the constant map  $\theta \mapsto 1$  tends to  $\psi_\mu$ .

Finally, the assertion concerning the regularity of the invariant curve is proved in the same way, the only change being the choice of the functional space : one takes the space of functions  $\psi : \mathbb{T} \rightarrow I$  of classe  $C^k$  with the  $k$ th derivative Lipschitz with Lipschitz constant less than 1.

**Exercise 3** *Such a space is complete when endowed with the  $C^0$  topology.*

This ends the proof of Theorem 2.

### 1.2.2 Dynamics on the invariant curves

In conclusion, from the “radial” hypothesis  $a_1(0) < 0$  we have obtained a complete control on the radial dynamics of  $F_\mu$  in a uniform neighborhood  $\mathcal{V}$  of 0 (i.e. figure 3 is still pertinent to describe the normal dynamics of  $F_\mu$ ), but we have no control of the dynamics restricted to the invariant curves. Indeed, this dynamics may be a “generic” dynamics of a diffeomorphism of the circle (see section 2). To be more precise we should add another “generic” assumption, this time on the “angular” part of  $F$ , namely that  $b_1(0) \neq 0$ , for example  $b_1(0) > 0$ . This implies that, for  $\mu$  close enough to 0, the restriction of the normal form  $N_\mu$  to its invariant circle  $C_\mu$  is a rotation whose angle increases with  $\mu$ . The two-parameter family  $f_{\omega,\mu}$  of diffeomorphisms of the circle defined by the restriction of  $F_\mu$  to its invariant curve  $\Gamma_\mu$ , the other parameter being  $\omega$ , behaves in general as does Arnold’s family  $T_{\omega,\mu}$ , introduced in [A3] (see section 2.8):

$$T_{\omega,\mu}(\theta) = \theta + \omega + \mu \cos 2\pi\theta.$$

### 1.2.3 A case of strong resonance

We give a hint of what happens in the case of a third order resonance  $\lambda = \lambda_0$  such that  $\lambda_0^3 = 1$ . The procedure used in the proof of Theorem 1 yields a local diffeomorphism

$$H : (\mathbb{C}, 0) \rightarrow (\mathbb{C}, 0), \quad \zeta \mapsto z = H(\zeta) = \zeta + O(|\zeta|^2)$$

such that

$$H \circ F \circ H^{-1}(z) = N(z) + O(|z|^3), \quad \text{where } N(z) = \lambda_0 z + c_0 \bar{z}^2.$$

The first resonant term dominates the non linear terms of the normal form which are invariant under rotation. Even if studying precisely the dynamics of  $F$  (or even of  $N$ ) does not look easy, it is possible to understand the main feature of the dynamics of perturbations, namely the appearance when  $c_0 \neq 0$ , in place of an invariant closed curve, of periodic points of order 3. It is natural to use as a parameter the eigenvalue  $\lambda \in \mathbb{C}$  in a neighborhood of  $\lambda_0$ . The same change of variables, depending now on  $\lambda$  yields a 2-parameter family of the form

$$G_\lambda(z) = H_\lambda \circ F_\lambda \circ H_\lambda^{-1}(z) = \lambda z + c(\lambda) \bar{z}^2 + O(|z|^3).$$

with  $c(\lambda_0) = c$ . A periodic orbit of period 3 of  $G_\lambda$  is made of fixed points of  $G_\lambda^3$ , that is of solutions  $z$  of

$$\begin{aligned} G_\lambda^3(z) - z &= (\lambda^3 - 1)z + (\lambda^2 + \bar{\lambda}^4 + \lambda \bar{\lambda}^2)c(\lambda) \bar{z}^2 + O(|z|^3) \\ &= 3\lambda_0^2 r e^{i(\beta+\varphi)} \left( \mu + C r e^{i(\alpha-\beta-3\varphi)} + O(\mu^2 + \mu r + r^2) \right) = 0, \end{aligned}$$

where we have noted

$$z = re^{i\varphi}, \quad \lambda - \lambda_0 = \mu e^{i\beta}, \quad c_0 = Ce^{i\alpha}, \quad \text{with } r, \mu, C \geq 0.$$

**Exercise 4** Using the implicit function theorem, show that for  $\lambda$  close to  $\lambda_0$  there are exactly three fixed points of  $G_\lambda^3$  near 0 and that these points tend to 0 when  $\lambda$  tend to  $\lambda_0$ . Indication: noting  $\varphi_0 = \frac{1}{3}(\alpha - \beta + (2k + 1)\pi)$  and fixing  $\beta$  and the integer  $k$ , set  $\mu = Cr(1 + \mu_1)$ ,  $\varphi = \varphi_0 + \varphi_1$  and look for a solution in the form  $r \mapsto (\mu_1(r), \varphi_1(r))$  of the equations in the neighborhood of  $r = 0$ ,  $\mu_1 = 0$ ,  $\varphi_1 = 0$ .

As suggested by Arnold in [A1], a more tractable way of studying  $F$  in the neighborhood of the origin is to consider it, after a local change variables, as a perturbation of the composition  $P_0(z) = \lambda_0 X^1(z)$  of the rotation  $z \mapsto \lambda_0 z$  with the time one map  $X^1$  of the differential equation  $(X) : \dot{z} = cz^2$ , which is invariant under this rotation. Figure 6 displays the dynamics of  $P_0$  when  $\lambda_0 = e^{\frac{2\pi i}{3}}$  and  $c$  is real: the lines  $\theta = \frac{2k\pi}{3}$ ,  $k = 0, 1, 2$  are cyclically permuted and the origin is neither an attractor nor a repellor. Next we deform  $P_0$  into the family

$$P_\epsilon(z) = \lambda_0 X_\epsilon^1(z), \quad \text{where } (X_\epsilon) : \dot{z} = \epsilon z + cz^2;$$

if  $\epsilon$  is real, the lines  $\theta = \frac{2k\pi}{3}$ ,  $k = 0, 1, 2$  are still cyclically permuted but, if  $\epsilon \neq 0$ , periodic points of order 3 appear on them.

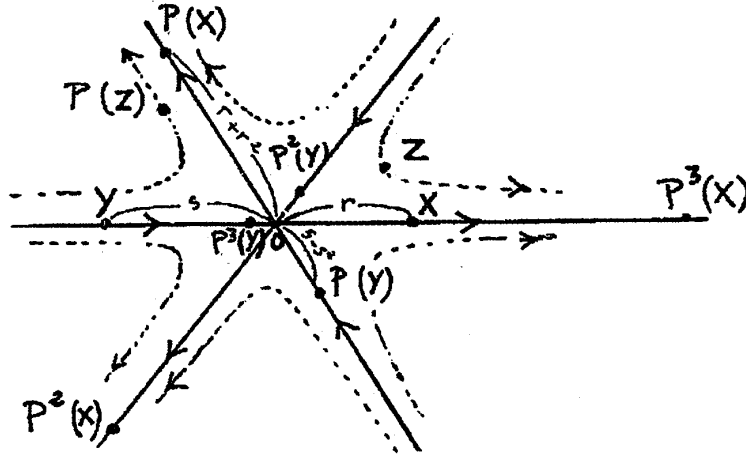


Figure 6. Dynamics of  $P$

Finally, figure 7 reproduces the phase portrait, given by Arnold in [A1], of the family (parametrized by  $\epsilon \in \mathbb{C}$ ) of vector-fields with the next term invariant under the rotation by  $2\pi/3$  added, that is:

$$\dot{z} = \epsilon z + cz^2 + Az|z|^2, \quad \text{Re}A < 0.$$



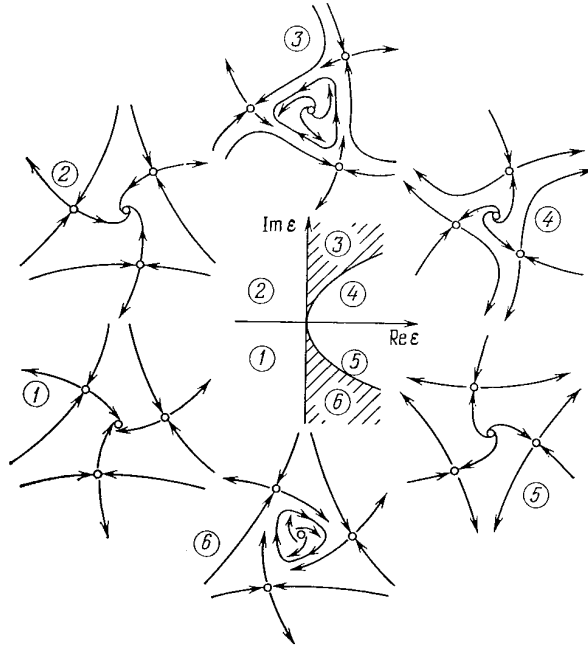


Fig. 7, borrowed from [A1].

**Remarks.** 1) The case  $\lambda^2 = 1$  is well understood. The case  $\lambda^4 = 1$ , say  $\lambda = i$ , is subtler because then a local change of variables can only achieve

$$H \circ F \circ H^{-1}(z) = N(z) + O(|z|^4), \quad \text{where } N(z) = \lambda(1 + a_1|z|^2)e^{2\pi i}(1/4 + b_1|z|^2) + c\bar{z}^3,$$

and there is a competition between the resonant terms of order 3 (See [A1, I]).

2) In the case of mappings, the phase portrait will be more complicated with in general; in particular, in some regions of the parameters, transverse intersections of stable and unstable manifolds of periodic points as on figure 8 will occur.

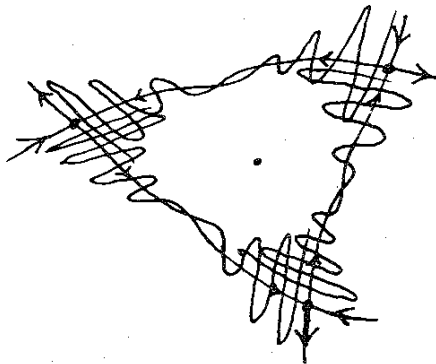


Fig. 8. Transverse intersection of invariant manifolds

**Exercise 5 (hard)** Supposing  $\lambda_0^3 = 1$ , compare as much as possible the dynamics of  $F(z) = \lambda_0 z + c_0 \bar{z}^2 + a_1 z|z|^2 + O(|z|^4)$  in a neighborhood of 0 to the one described in Figure 7.

**Exercise 6** Instead of solving directly the equation  $F_\lambda^3(z) = z$ , try, as is done in [I], to solve the system

$$F_\lambda(z_1) = z_2, \quad \lambda(z_2) = z_3, \quad F_\lambda(z_3) = z_1.$$

*Indication.* It is helpful to use the following variables:

$$y_1 = x_1 + x_2 + x_3, \quad y_2 = x_1 + \lambda_0 x_2 + \lambda_0^2 x_3, \quad y_3 = x_1 + \lambda_0^2 x_2 + \lambda_0^4 x_3.$$

The advantage is that the periodic orbits of  $F_\lambda$  will naturally coincide with orbits of a rotation of order 3.

**Exercise 7 (Normal forms of vector-fields in the neighborhood of a zero)**

We consider now a differential equation

$$\begin{aligned} \frac{dX}{dt} &= AX + F(X), \quad X = (x_1, \dots, x_n) \in \mathbb{R}^n, \quad A \in L(\mathbb{R}^n, \mathbb{R}^n), \\ F &= (f_1, \dots, f_n), \quad f_i(X) = O(\|X\|^2), \quad i = 1, \dots, n. \end{aligned}$$

Let  $H : \mathbb{R}^n, 0 \rightarrow \mathbb{R}^n, 0$  be a local diffeomorphism of the form

$$Y = H(X) = X + h(X), \quad h = (h_1, \dots, h_n),$$

where the  $h_i$  are homogeneous polynomials of the same degree  $r \geq 2$ . Show that the transformed equation is

$$\frac{dY}{dt} = AY + F(Y) + [A, h](Y) + O(\|Y\|^{r+1}), \quad \text{where } [A, h](Y) = Dh(Y)AY - Ah(Y).$$

When  $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ , show that the kernel of the operator  $h \mapsto [A, h]$  is generated by terms of the form  $h = (h_1, \dots, h_n)$  such that

$$h_i = 0 \text{ if } i \neq s, \quad h_s(Y) = y^{i_1} \dots y^{i_n}, \quad \text{with } i_1 + \dots + i_n = s, \quad i_1 \lambda_1 + \dots + i_n \lambda_n - \lambda_s = 0.$$

As in the case of diffeomorphisms, relations  $i_1 \lambda_1 + \dots + i_n \lambda_n - \lambda_s = 0$  are called resonance relations. Conclude to the existence of normal forms of the differential equation at any order (or even formal), where only resonant terms can be different from zero. Study in particular the following two cases:

- 1) a differential equation in the plane  $\mathbb{C} = \mathbb{R}^2$  of the form  $\frac{dz}{dt} = i\omega z + F(z, \bar{z})$ ;
- 2) a differential equation in  $\mathbb{C}^2 = \mathbb{R}^4$  of the form

$$\frac{dz_1}{dt} = i\omega_1 z + F_1(z_1, z_2, \bar{z}_1, \bar{z}_2), \quad \frac{dz_2}{dt} = i\omega_2 z + F_2(z_1, z_2, \bar{z}_1, \bar{z}_2)$$

when there exists integers  $l, m$  such that  $l\omega_1 = m\omega_2$ .

### 1.3 The area preserving case

When  $F$  is area preserving, the dynamics, being dictated by the angular behavior, is more intricate. Coming back to theorem 1, we notice that the radial part  $1 + f(|z|^2)$  of the normal form must be trivial, that is  $f \equiv 0$ ; indeed if not, the image by  $F$  of a circle close enough to the fixed point would be contained in the interior or the exterior of the disc bounded by this circle, which would contradict the preservation of area. We start proving the

**Proposition 4** *If, in addition to the hypotheses of theorem 1, the local diffeomorphism  $F$  is area preserving, the local change of coordinates  $H$  can be chosen area preserving.*

*Proof.* The proof is by induction: there are two cases according to whether the induction hypothesis is that there exists an area preserving change of coordinates  $H$  such that (see theorem 1)

$$H \circ F \circ H^{-1}(z) = N_n(z) + O(|z|^{2n+2}), \text{ (resp. } O(|z|^{2n+3}),$$

$$\text{where } N_n(z) = ze^{2\pi i(\omega + g_n(|z|^2))}.$$

We start with the first case; performing this change of coordinates and calling again  $F$  (instead of  $H \circ F \circ H^{-1}$ ) the result, that is  $F(z) = N_n(z) + O(|z|^{2n+2})$ , we notice that  $F \circ N_n^{-1}$  being composed of two area preserving local diffeomorphisms, is itself area preserving. Writing

$$F \circ N_n^{-1}(z) = z + R(z, \bar{z}) + O(|z|^{2n+3}),$$

where  $R(z, \bar{z})$  is a homogeneous polynomial of degree  $2n + 2$ , we get

$$F(z) = N_n(z) + R(\lambda z, \bar{\lambda} \bar{z}) + O(|z|^{2n+3}).$$

Now, if  $\lambda^{2n+3} \neq 1$ , theorem 1 asserts the existence of a unique local diffeomorphism  $H_{n+1} = Id + h$ , with  $h$  homogeneous of degree  $2n + 2$  such that

$$H_{n+1} \circ F \circ H_{n+1}^{-1}(z) = N_n(z) + O(|z|^{2n+3}).$$

By a computation similar to the one in the proof of theorem 1, the condition on  $h$  is

$$\lambda h(z, \bar{z}) - h(\lambda z, \bar{\lambda} \bar{z}) = R(\lambda z, \bar{\lambda} \bar{z}).$$

Now, the fact that  $F \circ N_n^{-1}$  is area preserving is equivalent to the determinant of its derivative being identically equal to 1. Writing

$$z = x + iy, \quad F \circ N_n^{-1}(z) = Z = X + iY, \quad R = A + iB,$$

and recalling that the derivative at *Identity* of the determinant function is the trace, we get the that  $\frac{\partial A}{\partial x}(x, y) + \frac{\partial B}{\partial y}(x, y) \equiv 0$ , which is equivalent to  $\frac{\partial R}{\partial z}(z)$  being purely real (recall that  $\frac{\partial R}{\partial z} = \frac{1}{2} \left[ \frac{\partial R}{\partial x} - i \frac{\partial R}{\partial y} \right]$ ).

**Lemma 5** Writing  $h = U + iV$ , one has  $\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \equiv 0$ . In other words, the local diffeomorphism  $H_n = Id + h$  preserves area at first order.

*Proof.* This is, as we know, equivalent to  $\frac{\partial h}{\partial z}(z, \bar{z})$  being purely imaginary for all  $z$ . But, taking the derivative of the equation satisfied by  $h$ , it follows only that

$$\frac{\partial h}{\partial z}(z, \bar{z}) - \frac{\partial h}{\partial z}(\lambda z, \bar{\lambda} \bar{z}) = \frac{\partial R}{\partial z}(\lambda z, \bar{\lambda} \bar{z}) \quad \text{is purely imaginary for all } z.$$

Nevertheless, this implies the conclusion: let

$$R(z, \bar{z}) = \sum_{i+j=2n+2} a_{ij} z^i \bar{z}^j, \quad h(z, \bar{z}) = \sum_{i+j=2n+2} b_{ij} z^i \bar{z}^j.$$

That  $\frac{\partial R}{\partial z}$  is purely imaginary is equivalent to

$$(i+1)a_{i+1,j} + (j+1)\bar{a}_{j+1,i} = 0.$$

Now, using the fact that  $h$  is such that  $\lambda^i \bar{\lambda}^j a_{ij} = (\lambda - \lambda^i \bar{\lambda}^j) b_{ij}$ , a direct computation shows that

$$(i+1)b_{i+1,j} + (j+1)\bar{b}_{j+1,i} = 0,$$

that is:  $\frac{\partial h}{\partial z}$  is purely imaginary, which is the conclusion.

Now comes the main point of the proof of proposition 4, which is inspired by the theory of generating functions in symplectic geometry (see [C2], section 3.3). We replace the local diffeomorphism  $H_n(x, y) = (x + U(x, y), y + V(x, y))$  which preserves area at first order only by

$$\tilde{H}_n(x, y) = (X, Y) = H_n(x, y) + O(|z|^{2n+3})$$

implicitly defined by the equations:

$$X = x + U(x, Y), \quad y = Y - V(x, Y).$$

The condition of preservation of area by  $\tilde{H}_n$  is

$$dX \wedge dY - dx \wedge dy = \left( \frac{\partial U}{\partial x}(x, Y) + \frac{\partial V}{\partial y}(x, Y) \right) dx \wedge dY = 0,$$

which is satisfied thanks to lemma 5.

In the second case, one starts with  $F(z) = N_n(z) + O(|z|^{2n+3})$ . If  $\lambda^{2n+4} \neq 1$ , theorem 1 asserts the existence of a local diffeomorphism  $H_{n+1} = Id + h$ , with  $h$  homogeneous of degree  $2n+3$  such that

$$H_{n+1} \circ F \circ H_{n+1}^{-1}(z) = N_{n+1}(z) + O(|z|^{2n+4}).$$

The difference with the first case is that  $H_{n+1}$  is not unique: resonant monomials  $b_{i+1,i} z^i |z|^i$  are not determined but only one choice is such that  $H_{n+1}$  preserves area at first order, namely choosing  $b_{i+1,i}$  purely imaginary.

### 1.3.1 Moser's invariant curve theorem ([M])

We now suppose that, in addition to satisfying  $\lambda^q \neq 1$  for all integers  $1 \leq q \leq 4$ . It follows that the radial component  $f$  of the normal form  $N$  vanishes identically and one can show that it is possible to choose  $H$  area preserving. Hence, one is reduced to the study in the neighborhood of its elliptic fixed point 0 of an area preserving diffeomorphism of  $\mathbb{C}, 0$  of the form

$$F(z) = N(z) + O(|z|^4), \quad N(z) = ze^{2\pi i(\omega + b_1|z|^2)}.$$

The normal form  $N$  is called a *truncated Birkhoff normal form*. Dynamically, it is an *integrable monotone twist*: as well as the rotation  $dF(0)$ , it leaves invariant each circle  $C_r$  centered at 0 but the angle of rotation  $2\pi(\omega + b_1r^2)$  on  $C_r$  varies now monotonically with the radius  $r$  of this circle

Poincaré, while studying the three body problem, became aware of a fundamental difference between the invariant circles on which  $N$  induces a periodic ( $\omega + b_1r^2$  rational) or non periodic ( $\omega + b_1r^2$  irrational) rotation: in the first case (angle  $2\pi\omega = 2\pi p/q$ ) the invariant circle is simply the union of a continuous family of  $q$ -periodic points  $z$  (i.e. of points  $z$  such that  $N^q(z) = z$ ); in consequence, a small perturbation should in general break such a circle, with only a finite number of periodic points surviving the perturbation (see more generally section 3). On the other hand, if  $\omega$  is irrational, the invariant circle being the closure  $\bigcup_{n \geq 0} N^n(z)$  of an orbit has a dynamical origin and hence has more chance to resist a perturbation. In the first volume of his famous book *The New Methods of Celestial Mechanics*, Poincaré even ventured to write that some arithmetic condition on  $\omega$  could perhaps grant resistance to perturbations of such an invariant circle but that he considered such a possibility as quite improbable (see [?] section 2.6).

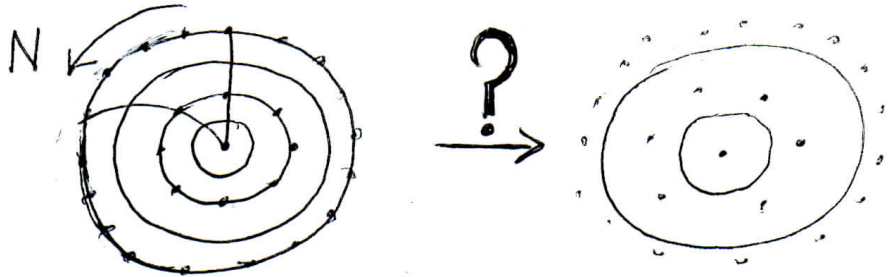


Figure 9. Perturbation of a monotone twist ???

Nevertheless, after the pioneering work of Kolmogorov in 1954, the so-called *KAM theory* (from the names of Kolmogorov, Arnold and Moser) showed that indeed, what Poincaré deemed improbable was in fact a dominant phenomenon. In the present case, the pertinent statement is the following

**Theorem 6 (Moser 1962)** *Given an area preserving diffeomorphism  $F$  as above, given  $C > 0$  and  $\beta > 0$ , there exists  $\epsilon(C, \beta) > 0$  such that each invariant circle  $C_{r_0}$  of the normal form  $N$  such that its rotation angle  $2\pi\omega_{r_0} = 2\pi(\omega + b_1 r_0^2)$  satisfies the diophantine condition*

$$\forall \frac{p}{q} \in \mathbb{Q}, \left| \omega_{r_0} - \frac{p}{q} \right| \geq \frac{C|\omega_{r_0} - \omega|}{|q|^{2+\beta}} \quad \text{and} \quad |\omega_{r_0} - \omega| < \epsilon(C, \beta)$$

*will give rise to a smooth (resp. analytic) closed curve  $\Gamma_{r_0}$  invariant under  $F$  and such that the restriction  $F|_{\Gamma_{r_0}}$  of  $F$  is smoothly conjugate to the rotation of angle  $2\pi\omega_{r_0}$ .*

The initial proof of theorem 6 by Moser in 1962 was refined by Rüssmann and Herman (see [He3]); the most transparent one (not the quickest one) is based on a version of the so-called “hard implicit function theorem” (see [Ham]) adapted to the problem of *small denominators* well known to astronomers since eighteenth century. The following consequence of area preservation, named *intersection property*, is the key point: the image  $F(\Gamma)$  of a curve  $\Gamma$  surrounding the origin cannot be disjoint from  $\Gamma$ . Note that such a property is preserved even under changes of coordinates which do not preserve area. Fixing  $r = r_0$  satisfying the hypotheses of the theorem, one chooses coordinates centered on  $C_{r_0}$  of the form:

$$z = r_0 \sqrt{1 + \sigma} e^{2\pi i \theta}.$$

The map  $F$  is now (as before we keep the same notation  $F$ )

$$F(\sigma, \theta) = (\sigma + O(r_0^4), \theta + \omega_{r_0} + b_1 r_0^2 \sigma + O(r_0^4)).$$

As a further simplification, one replaces  $\sigma$  by  $\rho = \sigma + O(r_0^2)$  so that the formula for  $F$  takes the form

$$F(\rho, \theta) = (\rho + \varphi(\rho, \theta), \theta + \omega_{r_0} + b_1 r_0^2 \rho),$$

where the perturbation  $\varphi$  is  $O(r_0^4)$ . Following Rüssmann, it is enough to look for a curve of the form  $\rho = \psi(\theta)$  which is sent by  $F$  to the *translated curve*  $\rho = \psi(\theta) + \tau$  for some  $\tau \in \mathbb{R}$ . This is because the intersection property, still valid after the changes of coordinates, implies that  $\tau$  must be equal to 0. This leads to the equation

$$\psi(g(\theta)) + \tau = \psi(\theta) + \varphi(\psi(\theta), \theta), \quad \text{where} \quad g(\theta) = \theta + \omega_{r_0} + b_1 r_0^2 \psi(\theta).$$

Recall that in the dissipative case, the radial hypothesis  $a_1(0) \neq 0$  implied the existence of a curve invariant under  $F_\mu$  with a prescribed normal dynamics. Having now an angular hypothesis  $b_1 \neq 0$ , it is natural to look for invariant curves of  $F$  with a prescribed angular dynamics. It turns out that the right constraint to impose to the (translated) curve we are looking for is the existence of a diffeomorphism  $h$  of the circle  $\mathbb{R}/\mathbb{Z}$  such that  $g(\theta) = h^{-1} \circ R_{\omega_{r_0}} \circ h(\theta)$ .

Finally, defining  $\psi$  by  $\psi(\theta) = \frac{1}{b_1 r_0^2} [h^{-1} \circ R_{\omega_{r_0}} \circ h(\theta) - \theta - \omega_{r_0}]$ , we must solve

$$\mathcal{F}(\varphi, \tau, h) := \psi(\theta) - \psi(h^{-1} \circ R_{\omega_{r_0}} \circ h(\theta)) - \tau + \varphi(\psi(\theta), \theta) = 0$$

in the neighborhood of the solution ( $\varphi = 0$ ,  $\tau = 0$ ,  $h = Id$ ). This is typically a “hard implicit function problem” (see [Ham]) because even the best diophantine condition allows us only to invert the “derivative” of  $\mathcal{F}$  in a weak sense (i.e. with loss of a finite number of derivatives on the target space of the inverse).

**Remark.** One can check ([K, C2]) that Moser’s invariant curve theorem applies to the Poincaré first return map on a surface of section of the planar circular restricted three body problem with any large enough energy (i.e. *Jacobi constant*) in the rotating frame. (see section 3.1) This implies stability in a strong sense as the invariant tori corresponding to the invariant closed curves are of codimension 1 in the energy surface and hence serve as barriers confining the solutions. This is precisely because he lacked such a theorem that Poincaré tried to prove such a stability result using barriers made from invariant manifolds of periodic orbits, which lead to the famous error in the first version of his prize winning Memoir on the Three-body problem (see [?]).

**Warning.** Examples in [AK] show that an area preserving  $C^\infty$ -diffeomorphism of the disk  $D^2$  with an elliptic fixed point such that  $\omega$  is a *Liouville number*, too well approximated by rational numbers, may have a very wild dynamics, with dense orbits.

## 2 Dynamics on the circle

Orientation preserving circle homeomorphisms (and diffeomorphisms) play a central role in the theory of dynamical systems. They appear naturally as return maps on a curve of section for differential equations without singular points on the 2-dimensional torus  $\mathbb{T}^2$ . Poincaré emphasizes that the problem posed by their study is simpler but reminiscent of problems which arise in Celestial Mechanics. He indeed introduced the main tool, the *rotation number*, which allows comparison with the simplest homeomorphisms of the circle, the rotations. Hence it is natural to start by studying the dynamics of rotations:

### 2.1 The dynamics of a rotation

**Exercise 8** 1) If  $\alpha = p/q \in \mathbb{Q}$ , any orbit of the rotation  $R_\alpha : x \mapsto x + \alpha$  from  $T^1 = \mathbb{R}/\mathbb{Z}$  to itself is periodic of minimal period  $q$  (the fraction  $p/q$  is supposed to be irreducible) and  $R_\alpha^q = Identity$ .

2) If  $\alpha$  is irrational, any orbit of  $R_\alpha$  is dense in  $T^1$

Hint for the proof of 2) Prove that the points of the orbit are two by two distincts and use compactity to infer that there is an accumulation point. Conclude by using the fact that any two rotations of the circle commute.

## 2.2 Lifting a homeomorphism of the circle to the real line

It is technically more convenient to deal with homeomorphisms of the real line, which means working in the universal cover  $D^0(\mathbb{T}^1)$  of the group  $\text{Homeo}_+(\mathbb{T}^1)$  of orientation preserving homeomorphisms of the circle:

**Lemma 7** *Each orientation preserving homeomorphism  $F : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  lifts to a homeomorphism  $f = \text{Id}_{\mathbb{R}} + \varphi : \mathbb{R} \rightarrow \mathbb{R}$  which is the sum of the Identity and a continuous 1-periodic function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  (which one identifies with a continuous function  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{R}$ ). This means that  $\pi \circ f = F \circ \pi$ , where  $\pi : \mathbb{R} \rightarrow \mathbb{T}^1$  is the canonical projection. Two such lifts  $f_1$  and  $f_2$  differ by an element  $z \in \mathbb{Z}$ , that is:  $f_2 = f_1 + z$ .*

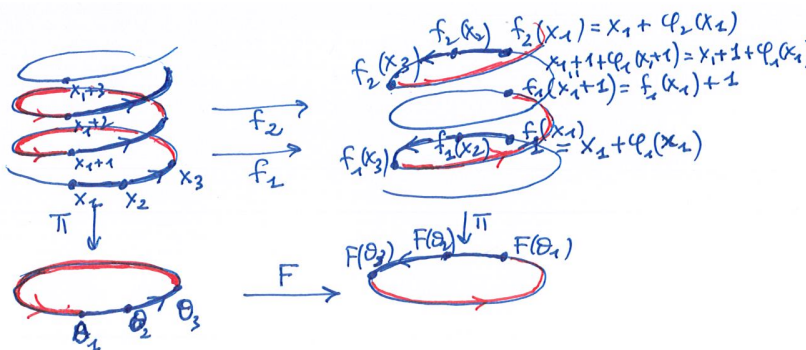


Figure 10. Lifting a homeomorphism of the circle.

*Proof.* The existence of  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that  $F \circ \pi = \pi \circ f$  follows formally from elementary homotopy theory because  $\mathbb{R}$  is contractible but it is essentially obvious on the figure. If  $f$  is a lift, so is  $x \mapsto f_z(x) := f(x + z)$  whatever be  $z \in \mathbb{Z}$ . Indeed,  $\pi \circ f_z(x) = \pi \circ f(x + z) = F \circ \pi(x + z) = F \circ \pi(x)$ . Moreover, being in the kernel of  $\pi$ , the difference  $f_z - f$  has values in  $\mathbb{Z}$  and hence is constant; as  $F$  is bijective and orientation preserving, this implies that  $f_1(x) = f(x + 1) = f(x) + 1$ . One concludes easily.

It follows from this lemma that the universal cover  $D^0(\mathbb{T}^1)$  of  $\text{Homeo}_+(\mathbb{T}^1)$  is

$$D^0(\mathbb{T}^1) = \{f : \mathbb{R} \rightarrow \mathbb{R}, f \text{ increasing homeomorphism}, f(x + 1) = f(x) + 1\}.$$

We endow  $D^0(\mathbb{T}^1)$  with the distance

$$d(f, g) = \max \left( \max_{x \in \mathbb{R}} |f(x) - g(x)|, \max_{x \in \mathbb{R}} |f^{-1}(x) - g^{-1}(x)| \right).$$

**Exercise 9** *Show that  $D^0(\mathbb{T}^1)$  is a topological group (i.e. that the maps  $(f, g) \mapsto f \circ g$  and  $f \mapsto f^{-1}$  are continuous) and that it is complete.*

**Exercise 10** *Prove the following proposition, which generalizes lemma 7 to arbitrary continuous maps and to higher dimensions in the following way:*



**Proposition 8** Any continuous map  $F : \mathbb{T}^r \rightarrow \mathbb{T}^r$  admits a lift  $f : \mathbb{R}^r \rightarrow \mathbb{R}^r$  such that  $\pi \circ f = F \circ \pi$ . Moreover, there exists a unique linear map  $\ell : \mathbb{R}^r \rightarrow \mathbb{R}^r$ , depending only on  $F$  and which is itself the lift of a map  $L : \mathbb{T}^r \rightarrow \mathbb{T}^r$ , such that any lift  $f$  of  $F$  is of the form  $f = \ell + \varphi$ , where for any  $z \in \mathbb{Z}^r$ ,  $\varphi(x+z) = \varphi(x)$  which means that  $\varphi$  may be identified with a map from  $\mathbb{T}^r$  to  $\mathbb{R}^r$ . In particular,  $F$  and  $L$  are homotopic.

### 2.3 Poincaré's rotation number

The rotation number of an orientation preserving homeomorphism of the circle was defined in 1885 by Poincaré in the third part [P] of his series of papers on the curves defined by differential equations.

We are interested in the behaviour of  $f \in D^0(\mathbb{T}^1)$  under iteration. One sees by induction that the  $k$ -th iterate  $f^k$  of  $f$  can be expressed as a so-called *Birkhoff sum*

$$f^n = Id + \varphi_n = Id + \sum_{i=0}^{n-1} \varphi \circ f^i.$$

**Theorem 9** For all  $f \in D^0(\mathbb{T}^1)$ , the sequence of periodic functions  $\frac{1}{n}(f^n - Id)$  converges uniformly when  $n \rightarrow \infty$  to a real number  $\rho(f) \in \mathbb{R}$  which is called the rotation number of  $f$ . It follows that  $\frac{1}{n}f^n$  converges to  $\rho(f)$  uniformly on compact subsets.

The map  $f \mapsto \frac{1}{n}(f^n - Id) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi \circ f^i$  being continuous from  $D^0(\mathbb{T}^1)$  to  $C^0(\mathbb{T}^1)$  both endowed with the  $C^0$  topology, and the convergence to  $\rho(f)$  being uniform, one deduces the important

**Corollary 10** The map  $f \mapsto \rho(f)$  is continuous from  $D^0(\mathbb{T}^1)$  endowed with the  $C^0$  topology to  $\mathbb{R}$ .

The key to the proof of theorem 9 is the following lemma, based on the existence of order and hence typical of dimension 1:

**Lemma 11** Let  $f = Id + \varphi \in D^0(\mathbb{T}^1)$ . Let  $m = \min_{x \in \mathbb{R}} \varphi(x)$ ,  $M = \max_{x \in \mathbb{R}} \varphi(x)$ . One has

$$0 \leq M - m < 1.$$

*Proof.* As  $\varphi$  is defined and continuous on the circle, there exist real numbers  $x_m$  and  $x_M$  such that

$$\varphi(x_m) = m, \quad \varphi(x_M) = M, \quad 0 \leq x_M - x_m < 1.$$

Because  $f$  is a homeomorphism which sends an interval of length 1 onto an interval of length 1, one has  $f(x_M) - f(x_m) < 1$ , i.e.  $M - m < 1 - (x_M - x_m) < 1$ .

**Corollary 12** For all  $x \in \mathbb{R}$ , the sequence

$$u_n = f^n(x) - x + 1, \quad n \geq 1,$$

is subadditive, that is:  $u_{n+m} \leq u_n + u_m$ .

*Proof.* Applying the lemma to  $f^n = Id + \varphi_n \in D^0(\mathbb{T}^1)$  and using the fact that  $f^n$  is increasing, we get

$$\forall x, y \in \mathbb{R}, \quad y - x - 1 \leq f^n(y) - f^n(x) \leq y - x + 1.$$

Taking  $y = f^m(x)$  we get

$$u_n + u_m - 2 \leq u_{n+m} \leq u_n + u_m.$$

The following lemma is classical

**Lemma 13** *If  $(u_n)_{n \geq 1}$  is a subadditive sequence in  $\mathbb{R} \cup \{-\infty\}$ , the sequence  $(\frac{u_n}{n})_{n \geq 1}$  converges in  $\mathbb{R} \cup \{-\infty\}$  and*

$$\lim_{n \rightarrow \infty} \frac{u_n}{n} = \inf_{n \geq 1} \frac{u_n}{n}.$$

*Proof.* Fix  $p \geq 1$ . For every  $n \geq p$ , write  $n = kp + r$ ,  $r < p$  and observe that

$$\frac{u_n}{n} \leq \frac{u_{kp} + u_r}{kp + r} \leq \frac{u_{kp}}{kp} + \frac{u_r}{kp + r} \leq \frac{u_p}{p} + \frac{u_r}{n}.$$

This implies that

$$\limsup_{n \rightarrow \infty} \frac{u_n}{n} \leq \frac{u_p}{p} \quad \text{and therefore} \quad \limsup_{n \rightarrow \infty} \frac{u_n}{n} \leq \inf_{p \geq 1} \frac{u_p}{p} \leq \liminf_{n \rightarrow \infty} \frac{u_n}{n}$$

*Proof of theorem 9.* From Corollary 12 and Lemma 13, one deduces that for all  $x \in \mathbb{R}$ , the sequence  $\frac{1}{n}(f^n(x) - x)$  converges in  $\mathbb{R} \cup \{-\infty\}$ . Moreover, the inequality used in the proof of the Corollary shows that the limit  $\rho(f)$  is independent of  $x$  and that the convergence is uniform. Also, from the inequalities  $u_n + u_m - 2 \leq u_{n+m} \leq u_n + u_m$ , one gets  $u_{n-1} + u_1 - 2 \leq u_n \leq u_{n-1} + u_1$  and by induction  $nu_1 - 2n \leq u_n \leq nu_1$ . It follows that  $u_1 - 2 \leq \lim_{n \rightarrow \infty} \frac{u_n}{n} \leq u_1$ , that is

$$\forall x \in \mathbb{R}, \quad f(x) - x - 1 \leq \rho(f) \leq f(x) - x + 1.$$

**Exercise 11** *Check that the same proof shows that the rotation number is still well defined if  $f = Id + \varphi$ , not necessarily being a homeomorphism, is non decreasing. The key is lemma 11 which does not hold in higher dimensions or even for general continuous degree one maps from the circle to itself (section 4).*

## 2.4 Rotation number and invariant measures

Let  $\mu$  be a probability measure on  $\mathbb{T}^1$  which is invariant under the homeomorphism  $\bar{f} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  and let  $f = Id + \varphi \in D^0(\mathbb{T}^1)$  be a lift of  $\bar{f}$ . We have, for any  $n \in \mathbb{N}$ ,

$$\mu(f^n - Id - n\mu(\varphi)) = \mu\left(\sum_{i=0}^{n-1} \varphi \circ f^i - n\mu(\varphi)\right) = \mu\left(\sum_{i=0}^{n-1} \varphi \circ \bar{f}^i\right) - n\mu(\varphi) = 0.$$

It follows that the function  $f^n - Id - n\mu(\varphi)$  must vanish somewhere. Applying lemma 11 to  $f^n \in D^0\mathbb{T}^1$  one gets that

$$\max(f^n - Id - n\mu(\varphi)) - \min(f^n - Id - n\mu(\varphi)) < 1,$$

and hence

$$|f^n - Id - n\mu(\varphi)|_{C^0} < 1.$$

This gives another proof of the uniform convergence of the sequence  $\frac{1}{n}(f^n - Id)$  to a constant. Summarizing, we have proved

**Proposition 14** *Let  $\bar{f} \in \text{Homeo}^+(\mathbb{T}^1)$  and let  $f = Id + \varphi \in D^0\mathbb{T}^1$  be a lift of  $\bar{f}$ . The rotation number  $\rho(f)$  of  $f$  satisfies  $\rho(f) = \mu(\varphi)$  for any  $\bar{f}$ -invariant probability measure on  $\mathbb{T}^1$ . One has*

$$\begin{cases} \forall n, \exists x_n \in \mathbb{R} & \text{such that } f^n(x_n) - x_n - n\rho(f) = 0, \\ \forall x \in \mathbb{R}, \forall n, & -1 < f^n(x) - x - n\rho(f) < 1. \end{cases}$$

Moreover, changing the lift  $f$  of  $\bar{f}$  does not change the class of  $\rho(f)$  in  $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ . This class is called the rotation number  $\rho(\bar{f})$  of  $\bar{f}$ .

Only the last part concerning the behaviour of  $\rho(f)$  under a change of the lift  $f$  remains to be proved: for this one notices that, as  $f^n(x+k) = f^n(x) + k$ , if  $g = f+k$ , one has  $g^n(x) = f^n(x) + nk$  and hence  $\rho(g) = \rho(f) + k$ . Alternatively, one notices that, as the total mas of  $\mu$  is 1,  $\mu(\varphi + k) = \mu(\varphi) + k$ .

**Corollary 15** *If  $p \in \mathbb{Z}$  and  $q \in \mathbb{N}, q \geq 1$ ,*

$$\begin{cases} \rho(f) = p/q \iff \exists x_q, f^q(x_q) = x_q + p, \\ \rho(f) > p/q \iff \forall x \in \mathbb{R}, f^q(x) > x + p, \\ \rho(f) < p/q \iff \forall x \in \mathbb{R}, f^q(x) < x + p, \end{cases}$$

*Proof.* If  $f^q(x_q) = x_q + p$ , one has also  $f^{kq}(x_q) = x_q + kp$  hence  $\rho(f) = \lim_{k \rightarrow \infty} \frac{1}{kq}(f^{kq}(x_q) - x_q) = p/q$ . If  $\rho(f) > p/q$ , one deduces from proposition 14 that  $\forall x \in \mathbb{R}, f^q(x) > x + q\rho(f) - 1 > x + p - 1$ . If for some  $x \in \mathbb{R}$  we have  $x + p - 1 < f^q(x) < x + p$ , the interval  $[x, x+1]$  is sent homeomorphically by  $f^q$  onto the interval  $[f^q(x), f^q(x) + 1]$ ; hence by the intermediate value theorem, there is some  $x_q \in [x, x+1]$  such that  $f^q(x_q) = x_q + p$  which implies that  $\rho(f) = p/q$ , a contradiction.

J.C. Yoccoz commented in a lecture that this Corollary gives a definition of the rotation number in the spirit of the definition of real numbers by Dedekind, while the definition by a limit is more in the spirit of Cauchy.

**Lemma 16** *A homeomorphism  $f \in D^0(\mathbb{T}^1)$  with rotation number  $\rho(f) = p/q$  is conjugate to the translation  $R_{p/q}$  if and only if  $f^q = R_p$ .*

*Proof.* If  $f = h^{-1} \circ R_{p/q} \circ h$  with  $h \in D^0(\mathbb{T}^1)$ , then  $f^q = h^{-1} \circ R_p \circ h = R_p$ . Conversely, if  $f^q = R_p$ , one checks that  $h = \frac{1}{q} \sum_{i=0}^{q-1} (f^i - i\frac{p}{q})$  belongs to  $D^0(\mathbb{T}^1)$  and conjugates  $f$  to  $R_{p/q}$ .

**Proposition 17 (Structure implied by a rational rotation number)** *Let  $\bar{f} \in \text{Homeo}_+(\mathbb{T}^1)$  be such that  $\rho(\bar{f}) = p/q \in \mathbb{Q}/\mathbb{Z}$  (irreducible). 1)  $\bar{f}$  has periodic points of period  $q$  and every periodic point of  $\bar{f}$  has minimal period  $q$ . 2) The limit sets  $\alpha(x)$  and  $\omega(x)$  of any element  $x \in \mathbb{T}^1$  are periodic orbits.*

*Proof.* Let  $f \in D^0(\mathbb{T}^1)$  be the lift of  $\bar{f}$  whose rotation number is  $p/q \in \mathbb{R}$ . It follows from Corollary 15 that  $f^q - p$  has a fixed point  $x_q$ , and hence that  $\bar{f}^q$  has a fixed point. If  $\bar{x}_{q'} \in \mathbb{T}^1$  is a periodic point of  $\bar{f}$  of period  $q'$ , it lifts to  $x_{q'} \in \mathbb{R}$  such that  $f^{q'}(x_{q'}) = x_{q'} + p'$ ; this implies that  $\rho(f) = p'/q' = p/q$ , hence that  $q' = kq$  which shows that  $x_{q'}$  is a periodic point of  $g = f^{q'} - p$  of period  $k$ . Now, the structure of elements  $g \in D^0(\mathbb{T}^1)$  whose rotation number is 0 is easily understood: the set  $\text{Fix}(g)$  of fixed points is closed and invariant under integer translations. If  $]a, b[$  is a connected component of  $\mathbb{R} \setminus \text{Fix}(g)$ , one deduces from the fact that  $g$  is increasing that if  $x \in ]a, b[$ ,  $\alpha(x) = a$  and  $\omega(x) = b$  (resp.  $\alpha(x) = b$  and  $\omega(x) = a$ ) if  $g - Id$  is positive (resp. negative) in the interval. This implies that the only periodic points are fixed points and proves also the last part of the proposition.

**Proposition 18 (Invariance under semi-conjugation)** *Let  $f, g \in D^0(\mathbb{T}^1)$  such that there exists a continuous map  $h = Id + \varphi \in C^0(\mathbb{T}^1, \mathbb{R})$  (i.e.  $\varphi$  continuous and 1-periodic) satisfying  $h \circ f = g \circ h$  (one says that  $f$  and  $g$  are semi-conjugated or that  $g$  is a factor of  $f$ ), then  $\rho(f) = \rho(g)$ .*

*Proof.* For any  $n$ ,  $h \circ f^n = g^n \circ h$ , hence  $f^n + \varphi \circ f^n = g^n \circ h$  and

$$\frac{1}{n}(f^n - Id) + \frac{\varphi \circ f^n}{n} = \frac{1}{n}(g^n - Id) \circ h + \frac{\varphi}{n},$$

hence the conclusion because  $\varphi$  is bounded. In particular, if  $f$  and  $g$  are conjugated by  $h \in D^0(\mathbb{T}^1)$ , i.e. if  $g = h \circ f \circ h^{-1}$ , they have the same rotation number  $\rho(f) = \rho(g)$ .

**Exercise 12** *If  $\bar{f}$  and  $\bar{g}$  are semi-conjugated in  $\text{Homeo}^+(\mathbb{T}^1)$ , that is if  $\bar{g} = \bar{h} \circ \bar{f} \circ \bar{h}^{-1}$ , there exist lifts  $f, g, h$  to  $D^0(\mathbb{T}^1)$  such that  $h \circ f = g \circ h$ .*

**Proposition 19** *If  $f, g \in D^0(\mathbb{T}^1)$  commute, then  $\rho(f \circ g) = \rho(f) + \rho(g)$ .*

*Proof.* It follows from ([C1], Corollary 17) that there exists a probability measure on  $\mathbb{T}^1$  which is invariant by both  $\bar{f}$  and  $\bar{g}$ . If  $f = Id + \varphi$  and  $g = Id + \psi$ , one has  $f \circ g = Id + \psi + \varphi \circ g$ , hence  $\rho(f \circ g) = \mu(\psi + \varphi \circ g) = \mu(\psi) + \mu(\varphi) = \rho(f) + \rho(g)$ .

**Exercise 13** *Show that if  $f, g \in D^0(\mathbb{T}^1)$  are lifts of two commuting elements  $\bar{f}, \bar{g} \in \text{Homeo}^+(\mathbb{T}^1)$ , they also commute.*

*Hint: use that, if  $\mu$  is a probability measure leaving both  $\bar{f}$  and  $\bar{g}$  invariant,  $\mu(f \circ g - Id) = \mu(g \circ f - Id)$ .*

**Proposition 20 (Structure implied by an irrational rotation number)**

Let  $\bar{f} \in \text{Homeo}_+(\mathbb{T}^1)$  be such that  $\rho(\bar{f}) \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ . 1) There exists a surjective continuous map  $\bar{h} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  such that  $\bar{h} \circ \bar{f} = R_{\rho(\bar{f})} \circ \bar{h}$ , i.e.  $\bar{f}$  is semi-conjugated to the corresponding rotation. 2) If  $\bar{f}$  is not actually conjugated to the corresponding rotation, there exists an invariant Cantor set  $X \subset \mathbb{T}^1$  which is the unique closed invariant minimal (for the inclusion) set. 3) Moreover  $X$  is at the same time the set  $\Omega(\bar{f})$  of non wandering points and the  $\alpha$ -limit set  $\alpha(x)$  and the  $\omega$ -limit set  $\omega(x)$  of every  $x \in \mathbb{T}^1$ .

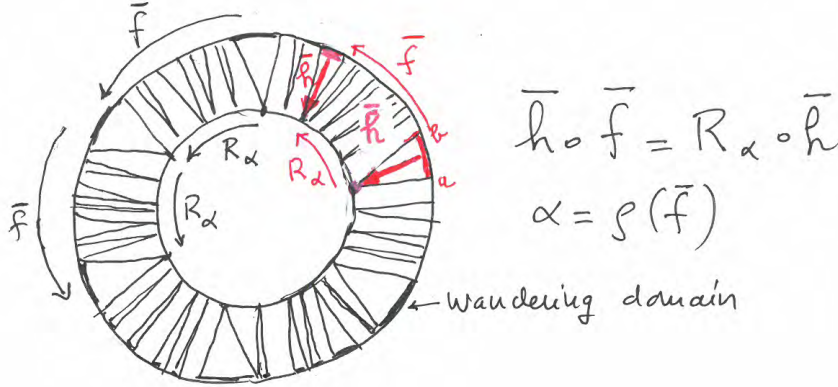


Figure 11. Typical behavior with irrational rotation number.

*Proof.* 1) Let  $\mu$  be a  $\bar{f}$ -invariant probability measure on  $\mathbb{T}^1$ . We shall still use the notation  $\mu$  for its lift to a positive Borel measure on  $\mathbb{R}$  invariant under integer translations (exercise: construct the lift); let  $f \in D^0(\mathbb{T}^1)$  be a lift of  $\bar{f}$ . Let

$$h(x) = \mu([0, x]).$$

As  $\bar{f}$  has no periodic point,  $\mu$  has no atomic mass hence  $h : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous non decreasing function such that  $h(x+n) = h(x)+n$  for any  $n \in \mathbb{Z}$ . Hence it defines a continuous surjective map  $\bar{h} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ . The  $\bar{f}$ -invariance of  $\mu$  implies that  $h(f(x)) - h(f(0)) = h(x) - h(0)$ , that is  $h \circ f = R_{h(f(0))-h(0)} \circ h$ . Finally, proposition 18 insures that  $h(f(0)) - h(0) = \rho(f)$ .

2) The map  $h$  is a homeomorphism if it is strictly increasing, that is if the support  $X$  of  $\mu$  is the whole circle  $\mathbb{T}^1$ . If not,  $X$  is a closed invariant set without isolated point (because  $\mu$  has no atomic mass) whose image by  $\bar{h}$  is  $\mathbb{T}^1$ ; moreover, the restriction  $\bar{h}|_X$  is injective except on the countable subset  $D \subset X$  formed by the extremities of the connected components of the complement  $\mathbb{T}^1 \setminus X$ . Let  $M \subset \mathbb{T}^1$  be a non empty closed  $\bar{f}$ -invariant set. Because  $\bar{h}$  is a semi-conjugation of  $\bar{f}$  to the rotation  $R_{\rho(\bar{f})}$ ,  $\bar{h}(M)$  is invariant under this rotation, hence  $\bar{h}(M) = \mathbb{T}^1$ ; as  $\bar{h}$  is injective on  $X \setminus D$  and non decreasing, this implies that  $M$  must contain  $X \setminus D$ . As  $X$  has no isolated point, the closure of  $X \setminus D$  is  $X$ , hence  $M \supset X$  which proves that  $X$  is the unique  $\bar{f}$ -invariant minimal closed set. It follows that  $X$  has no interior (otherwise its boundary would be invariant, contradicting minimality of  $X$ ), hence  $X$  is Cantor set.

3) Let  $I_0$  be a connected component of  $\mathbb{T}^1 \setminus X$ . Its iterates  $\bar{f}^n(I_0)$  are also connected components of  $\mathbb{T}^1 \setminus X$  and hence are two by two disjoint because  $\bar{f}$  has no periodic point. This means that any  $x \in \mathbb{T}^1 \setminus X$  is wandering, hence that  $\Omega(\bar{f})$  is contained in  $X$  and hence equal to it because  $X$  is minimal. Finally, being minimal,  $\Omega(\bar{f})$  coincides with  $\omega(x)$  and  $\alpha(x)$  for any  $x \in \mathbb{T}^1$ .

## 2.5 Unique ergodicity and its consequence

**Lemma 21** *If  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , the Haar measure is the only probability measure which is invariant under the rotation  $R_\alpha : \mathbb{T}^1 \rightarrow \mathbb{T}^1$ .*

*Proof.* The Haar measure  $\mu$  on  $\mathbb{T}^1$  is the direct image  $\pi_*\lambda$  of the Lebesgue measure on the unit interval  $[0, 1]$ . The Riesz representation theorem insures that it is well defined by its values on continuous functions  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{C}$  :

$$\int_{\mathbb{T}^1} \varphi d\mu = \int_{[0,1]} \varphi \circ \pi d\lambda,$$

where  $\pi : [0, 1] \rightarrow \mathbb{T}^1 = [0, 1]/(0 = 1)$  is the quotient projection.

1) As the Lebesgue measure is invariant under any translation of  $\mathbb{R}$ , this measure is invariant under any rotation of  $\mathbb{T}^1$  and it is the only one with this property. The proof is an easy exercise which consists in considering finer and finer partitions of  $\mathbb{T}^1$  into equal pieces which are obtained from one of them by rotations, for example the half-open intervals

$$C_k^m = \pi \left( \left[ \frac{k}{2^m}, \frac{k+1}{2^m} \right] \right),$$

and approach uniformly any continuous function  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{C}$  by a family of functions  $\varphi_m$  constant on the pieces of such partitions. One concludes because two rotation-invariant probability measures take the same value on the  $\varphi_m$ .

2) To prove that the Haar measure is the only probability measure which is invariant under the single “irrational” rotation  $R_\alpha$ , we shall use exercise 8: let  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{C}$  be a continuous function and let  $\mu$  be a  $R_\alpha$ -invariant measure. For any  $\theta \in \mathbb{T}^1$ , there exists a sequence  $(n_k)_{k \in \mathbb{N}}$ , which tends to  $+\infty$  such that  $\lim_{k \rightarrow \infty} R_\alpha^{n_k} = R_\theta$ . As  $\varphi$  is uniformly continuous, one deduces that

$$\int_{\mathbb{T}^1} \varphi(R_\theta x) d\mu(x) = \lim_{k \rightarrow \infty} \int_{\mathbb{T}^1} \varphi(R_\alpha^{n_k} x) d\mu(x) = \int_{\mathbb{T}^1} \varphi(x) d\mu(x).$$

Hence  $\mu$  is invariant under the full group of rotations of  $\mathbb{T}^1$  and one concludes by 1).

This property of admitting a unique invariant probability measure is called *unique ergodicity* (see [C1], section 5.4).

**Proposition 22 (Carleman, Denjoy, Furstenberg)** *If  $\bar{f} \in \text{Homeo}_+(\mathbb{T}^1)$  has an irrational rotation number, it is uniquely ergodic.*

*Proof.* By lemma 21, a rotation with irrational rotation number is uniquely ergodic and by proposition 20, there exists a semi-conjugation  $\bar{h}$  of  $\bar{f}$  to the rotation  $R_\rho(\bar{f})$ . Let  $S \subset \mathbb{T}^1$  be the set of points  $x$  such that  $\bar{h}^{-1}(x)$  is an interval.  $S$  is countable hence of Haar measure 0. On the other hand, any  $\bar{f}$ -invariant probability measure  $\mu$  satisfies  $\mu(\bar{h}^{-1}(S)) = 0$  because the wandering open intervals are disjoint, hence they have measure 0, and their boundaries are countable, hence also of measure zero because the absence of periodic points implies that  $\mu$  has no atoms. Finally,  $\bar{h} : \mathbb{T}^1 \setminus \bar{h}^{-1}(S) \rightarrow \mathbb{T}^1 \setminus S$  is a bimeasurable bijection, which proves that  $\bar{h} : (\mathbb{T}^1, \mu) \rightarrow \mathbb{T}^1, \text{Haar}$  is an isomorphism of measured space, which defines uniquely the measure  $\mu$ .

The following characterization of unique ergodicity makes it very useful:

**Proposition 23** *Let  $T : X \rightarrow X$  be a continuous map of a compact topological space into itself. The following conditions are equivalent:*

- i) *There is a unique probability measure on  $X$  which is invariant under  $T$ .*
- ii) *For any continuous function  $f : X \rightarrow \mathbb{C}$ , the sequence  $\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$  converges uniformly to a constant function.*

*Proof.* Suppose there is a unique  $T$ -invariant probability measure  $\mu$  and suppose by contradiction that there is a real number  $\epsilon > 0$ , a sequence  $(n_k)_{k \geq 0}$  of integers tending to  $+\infty$  and a sequence  $(x_k)_{k \geq 0}$  of points of  $X$  such that for all  $k$ ,

$$\left| \frac{1}{n_k} \sum_{i=0}^{n_k-1} f \circ T^i(x_k) - \int f d\mu \right| > \epsilon.$$

Let  $\mu_k = \frac{1}{n_k} \sum_{i=0}^{n_k-1} T_*^i \delta_{x_k}$ , where  $\delta_{x_k}$  is the Dirac measure at  $x_k$ . By compactity of the space  $\mathcal{M}(X)$  of Borel probability measures on  $X$  in the *weak\** topology (see [C1], beginning of section 3), one can suppose that the sequence  $(\mu_k)_{k \geq 0}$  converges weakly to a probability measure  $\mu'$  which is  $T$ -invariant because

$$\|T_* \mu_k - \mu_k\| = \left\| \frac{1}{n_k} (T_*^{n_k} \delta_{x_k} - \delta_{x_k}) \right\| \leq \frac{2}{n_k}.$$

This contradicts unicity because  $\int f d\mu' = \lim_{k \rightarrow \infty} \int f d\mu_k \neq \int f d\mu$ . For the converse, if  $L(f)$  is the uniform limit of the sequence  $\frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i$ , each  $T$ -invariant probability measure  $\mu$  satisfies

$$L(f) = \int L(f) d\mu = \lim_{n \rightarrow \infty} \int \frac{1}{n} \sum_{i=0}^{n-1} f \circ T^i d\mu = \frac{1}{n} \sum_{i=0}^{n-1} \int f d\mu = \int f d\mu.$$

**Corollary 24** *Let  $f$  be a  $C^1$  diffeomorphism of  $\mathbb{T}^1$  with an irrational rotation number and let  $\mu$  be its unique invariant probability measure. Then*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \log Df^n = \int_{\mathbb{T}^1} \log(Df) d\mu = 0.$$

*In words, the derivative of the iterates of  $f$  has at most a subexponential growth.*

*Proof.* As  $\frac{1}{n} \log Df^n$  coincides with the Birkhoff sum  $\frac{1}{n} \sum_{k=0}^{n-1} \log Df \circ f^k$ , it converges uniformly to  $\int_{\mathbb{T}^1} \log(Df) d\mu$ . This implies the result because,  $f^n$  being a diffeomorphism of  $\mathbb{T}^1$ , its average  $\int_{\mathbb{T}^1} Df^n = +1$  (if  $f$  is orientation preserving) but if  $\int_{\mathbb{T}^1} \log(Df) d\mu$  was strictly positive (resp. strictly negative),  $Df^n$  would converge uniformly to  $+\infty$  (resp. to 0), which would be a contradiction. It is a strengthening of this corollary, valid when slightly strengthening the  $C^1$  hypothesis on  $f$ , which plays the main part in the proof of Denjoy's theorem 25 given in the next section. Notice that the argument in the proof of this corollary makes critical use of the fact that the dimension is 1.

## 2.6 Denjoy's theorem

The behavior depicted in Proposition 20, is typical for homeomorphisms and can occur even for some  $C^1$  diffeomorphisms (Denjoy examples, see 2.7) but it cannot occur as soon as a little more regularity of the map is granted. More precisely:

**Theorem 25 (Denjoy [D])** *A  $C^1$  diffeomorphism of the circle  $\mathbb{R}/\mathbb{Z}$  whose derivative has bounded variation, and whose rotation number is irrational, is topologically conjugate to the corresponding rotation. In particular, it has no wandering domains.*

*Proof.*

According to Proposition 20 it is enough to show that  $f$  has no wandering domain. Let us suppose by contradiction that there exists one i.e. that there exists an open subset of the circle which is the union of a countable set of disjoint intervals  $I_n$  such that  $f$  sends  $I_n$  to  $I_{n+1}$ . Then for any  $i \in \mathbb{Z}$ ,  $f^i$  sends  $I_0$  to  $I_i$  and  $I_{-i}$  to  $I_0$ . As  $\sum_{i \in \mathbb{Z}} l(I_i) < 1$  (where  $l(I_i)$  is the Lebesgue measure on  $\mathbb{T}^1$ , that is the length of any lift to  $\mathbb{R}$  of  $I_i$ ), we have that  $\lim_{i \rightarrow \pm\infty} l(I_i) = 0$ . Hence we get a contradiction as soon as we can find a sequence  $q_n$  of integers and a positive constant  $K$  such that

$$q_n \rightarrow \infty, \quad \text{and} \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{T}^1, |\log Df^{q_n}(x)| < K.$$

This will follow from applying the following proposition to  $\varphi = \log Df$ :

**Proposition 26 (Denjoy-Koksma inequality)** *Let  $f \in D^0(\mathbb{T}^1)$  with rotation number  $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$  and let  $\frac{p}{q} \in \mathbb{Q}$  be irreducible and such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

*Let  $\varphi : \mathbb{T}^1 \rightarrow \mathbb{R}$  be continuous with bounded variation and let  $\mu$  be a  $\bar{f}$ -invariant probability measure on  $\mathbb{T}^1$ . One has*

$$\sup_{x \in \mathbb{T}^1} \left| \sum_{i=0}^{q-1} \varphi(\bar{f}^i(x)) - q\mu(\varphi) \right| < \text{Var}(\varphi).$$



Indeed, this inequality implies the contradiction we are looking for thanks to the two following assertions:

1) In proposition 26, if  $\varphi = \log Df$ , one has  $\mu(\varphi) = 0$ .

2)  $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$ ,  $\exists p_n, q_n \rightarrow +\infty$  such that  $\forall n \in \mathbb{N}$ ,  $\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}$ ;

Assertion 1) is just corollary 24.

The proof of assertion 2) is classical: the most direct one is by using the so-called Dirichlet's drawers principle: "if one has more socks than drawers, at least one drawer contains two socks". Here the socks are the  $Q + 1$  elements  $0, \alpha, 2\alpha, \dots, Q\alpha \in \mathbb{T}^1$  and the drawers are the  $Q$  intervals  $[0, \frac{1}{Q}], [\frac{1}{Q}, \frac{2}{Q}], \dots, [\frac{Q-1}{Q}, 1]$ ,  $Q$  being an arbitrary integer. Hence there exists integers  $q_1, q_2 \leq Q, p$  such that  $|q_1\alpha - q_2\alpha - p| < \frac{1}{Q}$ . Setting  $q = |q_1 - q_2|$  we have  $q \leq Q$ , hence  $|q\alpha - p| < \frac{1}{Q} \leq \frac{1}{q}$ . As  $Q$  is arbitrary, the proof is complete.

A more informative proof is given by taking as  $\frac{p_n}{q_n}$  the convergents of the continued fraction representation of  $\alpha$  (see [Khl]).

*Proof of proposition 26.*

**Lemma 27** *Let  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and  $\frac{p}{q} \in \mathbb{Q}$  be irreducible and such that  $\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}$ . Setting  $\alpha_i = i\alpha \pmod{1}$ ,  $0 \leq \alpha_i < 1$ , there is exactly one of the  $\alpha_i$ ,  $i = 1, \dots, q$  in each interval  $]\frac{k}{q}, \frac{k+1}{q}[$ ,  $0 \leq k \leq q - 1$ .*

*Proof.* Let us suppose that  $0 < \alpha - \frac{p}{q} < \frac{1}{q^2}$  (the case  $-\frac{1}{q^2} < \alpha - \frac{p}{q} < 0$  is similar). If  $i = 1, \dots, q$ , we have

$$0 < i\alpha - i\frac{p}{q} < \frac{i}{q^2} \leq \frac{1}{q}.$$

As the fraction  $\frac{p}{q}$  is irreducible, the  $\frac{ip}{q} \pmod{1}$  are all distinct: indeed, if  $\frac{ip}{q} - \frac{jip}{q} = k \in \mathbb{Z}$ , we have  $(i - j)p = kq$  which contradicts irreducibility. The conclusion follows from the fact that, for all  $i$ , the point  $\alpha_i$  is at less than  $\frac{1}{q}$  at the right of  $\frac{ip}{q} \pmod{1}$ .

We are ready to estimate the term  $\left| \sum_{i=1}^{q-1} \varphi(\bar{f}^i(x)) - q\mu(\varphi) \right|$  in proposition 26.

Recall the semi-conjugation  $\bar{h} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  defined in Proposition 20. Starting from  $x \in \mathbb{T}^1$  arbitrary, we choose  $q$  points  $y_0 = x, y_1, y_2, \dots, y_{q-1}$  circularly ordered on  $\mathbb{T}^1$  and such that  $\forall i = 0, 1, \dots, q - 1$ ,

$$\bar{h}(y_i) = \frac{i}{q} + h(x), \text{ that is } \int_{y_{k_i}}^{y_{k_i+1}} d\mu(t) = \frac{1}{q}.$$

Such a choice is a priori not unique, as explained on figure 12.

We have

$$\begin{aligned}
 \left| \sum_{i=0}^{q-1} \varphi(\bar{f}^i(x)) - q\mu(\varphi) \right| &= \left| \sum_{i=0}^{q-1} \left[ \varphi(\bar{f}^i(x)) - q \int_{y_{k_i}}^{y_{k_i+1}} \varphi(t) d\mu(t) \right] \right| \\
 &= \left| \sum_{i=0}^{q-1} q \int_{y_{k_i}}^{y_{k_i+1}} [\varphi(\bar{f}^i(x)) - \varphi(t)] d\mu(t) \right| \\
 &\leq \sum_{i=0}^{q-1} q \left| \int_{y_{k_i}}^{y_{k_i+1}} [\varphi(\bar{f}^i(x)) - \varphi(t)] d\mu(t) \right| \\
 &\leq \sum_{i=0}^{q-1} q \times \frac{1}{q} \times \sup_{t \in [y_{k_i}, y_{k_i+1}]} |\varphi(\bar{f}^i(x)) - \varphi(t)| \\
 &\leq \text{Var}(\varphi).
 \end{aligned}$$

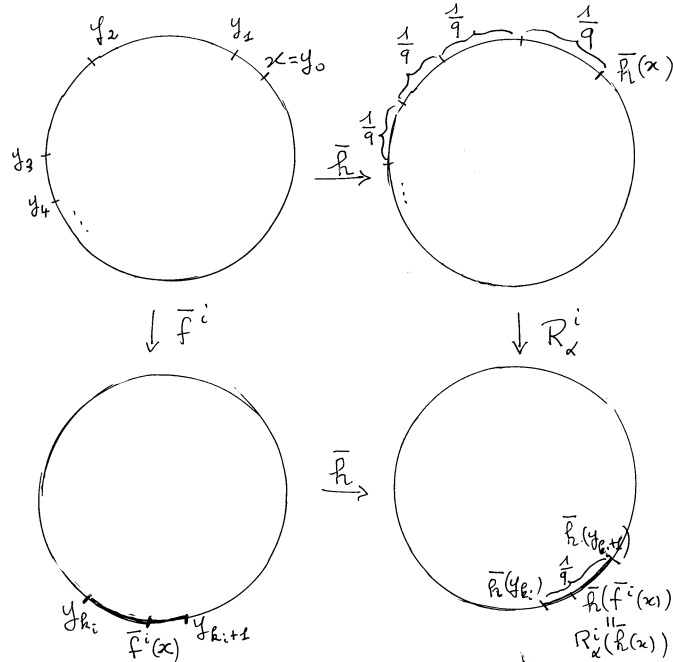


Figure 12. Choice of the  $y_i$ .

**Exercise 14** Show that the space of functions on  $\mathbb{T}^1$  with bounded variation is dense in the space  $C^0(\mathbb{T}^1)$  of continuous functions endowed with the topology of uniform convergence. Notice then that Proposition 26 implies unicity of the invariant probability measure  $\mu$  of a homeomorphism of  $\mathbb{T}^1$  whose rotation number is irrational.

**Remark.** One remarkable feature of Denjoy's theorem is that, if the rotation number  $\alpha$  is too well approximated by rationals, the conjugation equation linearized at the rotation with rotation number  $\alpha$  has not always a solution. More

precisely, let us write  $f(x) = x + \alpha + \delta f(x)$  and look for a diffeomorphism  $h(x) = x + \delta h(x)$  such that  $h \circ f = R_\alpha \circ h$ , that is

$$x + \alpha + \delta f(x) + \delta h(x + \alpha + \delta f(x)) = x + \alpha + \delta h(x).$$

Forgetting terms of order at least 2 in  $\delta f$ ,  $\delta h$  and their derivatives, we get the so-called *homological equation*

$$\delta h(x + \alpha) - \delta h(x) = -\delta f(x).$$

Supposing  $\delta f$  of class  $C^r$ , and writing its Fourier series  $\delta f(x) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k x}$ , we get that  $a_0$  must vanish and the Fourier series  $\sum_{k \in \mathbb{Z}} b_k e^{2\pi i k x}$  of  $\delta h$  must satisfy

$$\forall k \in \mathbb{Z} \setminus 0, b_k = \frac{a_k}{e^{2\pi i k \alpha} - 1}.$$

The condition  $a_0 = 0$  would be taken care of by the condition that the rotation number of  $f$  be  $\alpha$  but for the other coefficients arises the *problem of small denominators*: if  $\alpha$  is too well approximated by rational numbers, they become arbitrarily large and the Fourier series of  $\delta h$  diverges. Nevertheless, for almost all  $\alpha$  in the sense of Lebesgue measure, the ones satisfying a so-called *diophantine hypothesis*

$$\exists C > 0 \text{ such that } \forall \frac{p}{q} \in \mathbb{Q}, \quad \left| \alpha - \frac{p}{q} \right| > \frac{C}{q^{2+\epsilon}}$$

(as in Theorem 6), the homological equation can be solved with only a finite loss of differentiability.

And indeed, a fundamental strengthening of this theorem has been given in Herman's thesis [He1] with further development by Yoccoz: under such a diophantine hypothesis on the rotation number, the conjugacy is  $C^{r-2}$  (resp.  $C^\infty$ , resp. analytic) if  $f$  is  $C^r$ ,  $r \geq 3$ , resp.  $C^\infty$ , resp. analytic). A local version, for  $f$  close to the corresponding rotation had been given first by Arnold in [A3].

## 2.7 Denjoy's $C^1$ counterexamples

[after [R]] An irrational number  $\alpha \in ]0, 1[$  being given, the problem is to construct a  $C^1$  diffeomorphism  $\bar{f}$  of  $\mathbb{T}^1$  whose dynamics is the one described on figure 11: one must construct a sequence of open intervals  $(I_n \subset \mathbb{T}^1)_{n \in \mathbb{Z}}$ , with total length  $\sum_{n \in \mathbb{Z}} l(I_n) = 1$  and such that once ordered as the orbits of the rotation  $R_\alpha$ , they satisfy  $\bar{f}(I_n) = I_{n+1}$ .

We start with a sequence  $(l_n)_{n \in \mathbb{Z}}$ , of positive real numbers which satisfy

$$\sum_{n \in \mathbb{Z}} l_n = 1 \quad \text{and} \quad \lim_{n \rightarrow \pm\infty} \frac{l_n}{l_{n+1}} = 1, \quad \text{for example } l_n = \frac{1}{(1+n^2) \sum_{m=-\infty}^{+\infty} \left( \frac{1}{1+m^2} \right)},$$

and, noting  $n\alpha \pmod{1} = \alpha_n \in [0, 1[$ , we define

$$I_n = [b_n, c_n], \quad b_n = \sum_{m, \alpha_m < \alpha_n} l_m, \quad c_n = \sum_{m, \alpha_m \leq \alpha_n} l_m.$$

On the one hand,  $l(I_n) = l_n$ , on the other hand, the  $I_n$  are disjoint and ordered as the orbits of the rotation  $R_\alpha$ : this follows from the obvious implication

$$(\alpha_n < \alpha_p) \Rightarrow (c_n < b_p).$$

Now, one must construct the restrictions  $f_n = \bar{f}|_{I_n} : I_n \rightarrow I_{n+1}$  in such a way that their extension by continuity to the whole circle be of class  $C^1$ . To get a  $C^0$  extension, linear maps  $f_n$  would be sufficient; in order to get  $C^1$  we must chose the  $f_n$  more carefully. Precisely, we chose their derivatives  $f'_n : I_n \rightarrow ]0, +\infty]$  so that

$$\left\{ \begin{array}{l} 1) \lim f'_n(t) = 1 \quad \text{if } t \rightarrow b_n \text{ and } t \rightarrow c_n, \\ 2) \lim f'_n(t) = 1 \quad \text{uniformly when } n \rightarrow \pm\infty, \\ 3) \int_{I_n} f'_n(s) ds = l_{n+1}. \end{array} \right.$$

Notice that condition 2) is realizable because  $\lim_{n \rightarrow \pm\infty} \frac{l_n}{l_{n+1}} = 1$ .

One extends continuously the  $f'_n$  to a map  $f' : \mathbb{R} \rightarrow ]0, +\infty[$  by setting

$$f'(t) = 1 \quad \text{for } t \in [0, 1[ \setminus (\cup_{n \in \mathbb{Z}} I_n) \quad \text{and} \quad f'(t+1) = f'(t),$$

and one defines  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) = b_1 + \int_0^t f'(s) ds.$$

By construction,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a  $C^1$  diffeomorphism such that  $f(t+1) = f(t) + 1$

**Lemma 28**

$$f(I_n) = \begin{cases} I_{n+1} & \text{if } \alpha_n < 1 - \alpha, \\ I_{n+1} \text{ translated by } +1 & \text{if } \alpha_n \geq 1 - \alpha. \end{cases}$$

*Proof.* As  $\int_{I_n} f'_n(s) ds = l_{n+1}$ , the assertion on  $f(c_n)$  follows from the one on  $f(b_n)$ . As  $\cup_{n \in \mathbb{Z}} I_n$  has full measure in  $\mathbb{T}^1$ , one has

$$f(b_n) = b_1 + \int_0^{b_n} f'(s) ds = b_1 + f\left(\sum_{m, \alpha_m < \alpha_n} l_m\right) = b_1 + \sum_{m, \alpha_m < \alpha_n} l_{m+1}.$$

1) If  $\alpha_n < 1 - \alpha$ , that is if  $\alpha_{n+1} = \alpha_n + \alpha$ ,

$$\{m, \alpha_m < \alpha_n\} = \{m, \alpha < \alpha_{m+1} < \alpha_{n+1}\}.$$

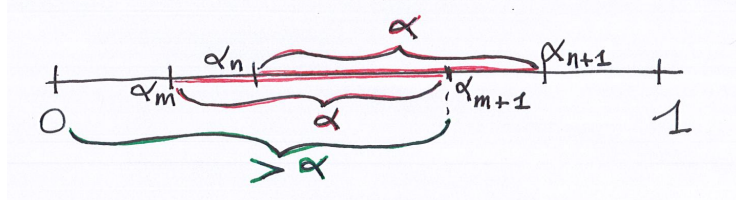


Figure 13.

Indeed, see figure 13,  $\alpha_m < \alpha_n$  implies  $\alpha_{m+1} = \alpha_m + \alpha < \alpha_n + \alpha = \alpha_{n+1}$ , hence  $\alpha_{m+1} > \alpha$ . Conversely, if  $\alpha_{m+1} < \alpha_{n+1}$  and  $\alpha_{m+1} > \alpha$ , then  $\alpha_m = \alpha_{m+1} - \alpha < \alpha_{n+1} - \alpha = \alpha_n$ . Finally,

$$f(b_n) = b_1 + \sum_{m, \alpha_{m+1} < \alpha_{n+1}} l_{m+1} - \sum_{m, \alpha_{m+1} < \alpha} l_{m+1} = b_1 + b_{n+1} - b_1 = b_{n+1}.$$

2) if  $\alpha_n > 1 - \alpha$ , that is if  $\alpha_{n+1} = \alpha_n + \alpha - 1$ ,

$$\{m, \alpha_m < \alpha_n\} = \{m, \alpha_{m+1} < \alpha_{n+1}\} \cup \{m, \alpha_{m+1} > \alpha\}.$$

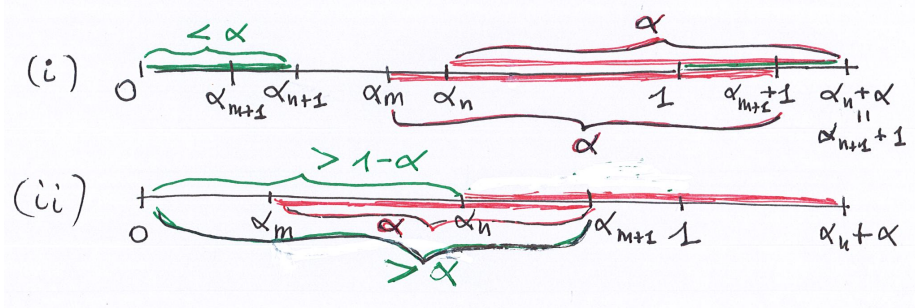


Figure 14.

Indeed, either  $\alpha_{m+1} < \alpha_{n+1} < \alpha$ , then  $\alpha_m = \alpha_{m+1} + 1 - \alpha < \alpha_{n+1} + 1 - \alpha = \alpha_n$  (Figure 14(i)) ; or  $\alpha_{m+1} > \alpha_{n+1} = \alpha_n + \alpha - 1$  ; in this case,  $\alpha_n = \alpha_{n+1} - \alpha + 1 < \alpha_{m+1} - \alpha + 1$ , hence  $\alpha_m < \alpha_n$  implies  $\alpha_m < \alpha_{m+1} - \alpha + 1$  hence that  $\alpha_{m+1} - \alpha = \alpha_m > 0$  ; conversely,  $\alpha_{m+1} > \alpha$  is equivalent to  $\alpha_m = \alpha_{m+1} - \alpha$  and implies  $\alpha_m < 1 - \alpha < \alpha_n$  (Figure 14(ii)). Finally,

$$f(b_n) = b_1 + \sum_{m, \alpha_{m+1} < \alpha_{n+1}} l_{m+1} + \sum_{m, \alpha_{m+1} \geq \alpha} l_{m+1} = b_1 + b_{n+1} + (1 - b_1) = b_{n+1} + 1.$$

It remains to prove that the rotation number of  $f$  is indeed  $\alpha$ : by induction,  $f(0) = b_1$  and  $f^n(0) = [n\alpha] + b_n$ , where  $[x] \leq x$  is the integral part of the real number  $x$ . Indeed,

$$\begin{aligned} f^{(n+1)}(0) &= f([n\alpha] + b_n) = [n\alpha] + f(b_n) \\ &= \begin{cases} [n\alpha] + b_{n+1} = [(n+1)\alpha] + b_{n+1} & \text{if } \alpha_n < 1 - \alpha, \\ [n\alpha] + b_{n+1} + 1 = [(n+1)\alpha] + b_{n+1} & \text{if } \alpha_n \geq 1 - \alpha. \end{cases} \end{aligned}$$

One concludes because this implies

$$|f^n(0) - n\alpha| = |b_n - \alpha_n| < 1.$$

### 2.7.1 Further refinements

- 1) There exist  $C^\infty$  homeomorphisms which are Denjoy counterexamples ([Ha])
- 2) There are no analytic homeomorphisms which are Denjoy counterexamples ([Y2])

## 2.8 The Arnold 1-parameter family

The analytic family of analytic diffeomorphisms  $f_{t,a} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  defined by

$$f_{t,a}(x) = x + a \sin(2\pi x) + t, \quad t \in \mathbb{R}, \quad 0 \leq a < \frac{1}{2\pi},$$

was introduced in [A3] and further studied in [He2]. Figure 15 shows the behavior of the rotation number  $\rho(f_{t,a})$  as a function of the parameters  $(t, a)$  : for  $a > 0$ , the function  $t \mapsto \rho(f_{t,a})$  is non decreasing and its graph is a *devil's staircase*.

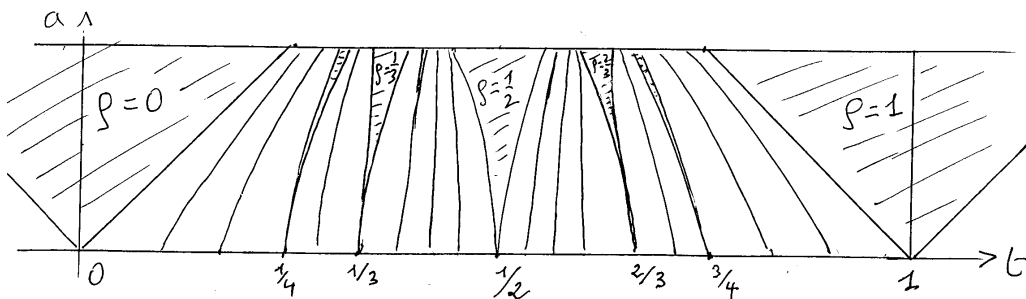


Figure 15. Arnold's tongues.

Moreover, for a fixed value of  $a$ , the set of  $t$  for which the rotation number of  $f_t$  is rational is big in the sense of topology, namely it is open and dense, but its complement is big in the sense of measure, namely, its measure tends to 1 when  $a \rightarrow 0$ . As already said in section 1.2.2, this example is a good illustration of the types of dynamics encountered in a “generic” family of analytic diffeomorphisms.

We now prove these assertions. Let us consider the partial order on functions from  $\mathbb{R}$  to  $\mathbb{R}$  defined by

$$f \geq g \quad \text{iff} \quad \forall x \in \mathbb{R}, f(x) \geq g(x).$$

**Lemma 29** *The map  $f \mapsto \rho(f)$  from  $D^0(\mathbb{T}^1)$  to  $\mathbb{R}$  is non decreasing and continuous.*

*Proof.* Both properties are direct consequences of the equality  $\rho(\text{Id} + \varphi) = \mu(\varphi)$  or of the definition of  $\rho$  as a uniform limit.

Let us call  $\rho_a : \mathbb{R} \rightarrow \mathbb{R}$  the function  $t \mapsto \rho(f_{t,a})$ .

**Proposition 30** *For each  $a \in ]0, \frac{1}{2\pi}[$  and each  $\frac{p}{q} \in \mathbb{Q}$ ,  $\rho_a^{-1}(\frac{p}{q})$  is a closed interval with a non empty interior.*

*Proof.* The fact that  $\rho_a^{-1}(\frac{p}{q})$  is connected is a direct consequence of the above lemma. The assertion about the interior is proved by contradiction: let us suppose that  $\rho_a^{-1}(\frac{p}{q}) = \{t_0\}$  is reduced to a single point; then, by Corollary 15,  $(f_{a,t}^q - R_p)(x)$  must be strictly negative for all  $x$  if  $t < t_0$  and strictly positive for all  $x$  if  $t > t_0$ , which implies that  $f_{a,t}^q - R_p$  must vanish identically. By lemma 16, this is equivalent to the fact that  $f_{a,t}$  is conjugated to  $R_{\frac{p}{q}}$ . That this is not the case follows from the next lemma:

**Lemma 31** *If  $a > 0$ ,  $f_{a,t}^q - R_p$  never vanishes identically.*

*Proof.* For each  $(a, t)$ ,  $f_{a,t}$  extends to an entire function  $f : \mathbb{C} \rightarrow \mathbb{C}$ . If the identity  $f_{a,t}^q - R_p$  holds in the real domain, it also holds for the extension, that is  $(f^q - R_p)(z) = 0$  for all  $z \in \mathbb{C}$ , or  $(R_{-p} \circ f^{q-1}) \circ f = Id_{\mathbb{C}}$ . Hence  $f$  should be a biholomorphic diffeomorphism of  $\mathbb{C}$ , that is of the form  $f(z) = \alpha z + \beta$  with  $\alpha \neq 0$ . But this implies  $a = 0$ .

This ends the proof of Proposition 30. The components of  $\cup_{a \geq 0} \rho_a^{-1}(\mathbb{Q})$  are called *Arnold's tongues*.

**Proposition 32** *If  $\rho_a(t_0) \in \mathbb{R} \setminus \mathbb{Q}$ , the function  $\rho_a$  is strictly increasing at  $t_0$ .*

*Proof.* By Denjoy's theorem 25, if  $\rho(f_{t,a}) = \alpha$  is irrational, there exists a homeomorphism  $h$  such that  $f_{t,a} = h \circ R_\alpha \circ h^{-1}$ . Let us suppose that  $\rho(f_{t',a}) = \alpha$ ; writing  $R_{t'-t} = f_{t',a} \circ (f_{t,a})^{-1}$ , we get  $t' - t = 0$  as an immediate consequence of the following lemma:

**Lemma 33** *Let  $f_1, f_2 \in D^0(\mathbb{T}^1)$  be such that  $\rho(f_1) = \rho(f_2) = \alpha \in \mathbb{R}$ . If  $f_2$  is  $C^0$ -conjugated to  $R_\alpha$ , then  $f_1 \circ f_2^{-1}$  has a fixed point.*

*Proof.* [of lemma 33] If  $f_2 = h \circ R_\alpha \circ h^{-1}$ , by conjugation, it is enough to show that  $(h^{-1} \circ f_1 \circ h) \circ R_\alpha^{-1}$  has a fixed point. In other words we are reduced to the case where  $f_2 = R_\alpha$ . But then, the conclusion follows from proposition 14.

**Remark.** Lemma 33 still holds without the hypothesis that  $f_2$  be  $C^0$ -conjugated to the rotation (see [He1] (4.1.1)).

From propositions 30 and 32 follows the

**Corollary 34** *If  $a > 0$ , the subset  $Int(\rho_a^{-1}(\mathbb{Q})) \subset \mathbb{R}$  is open and dense (and hence its complement is a perfect and totally discontinuous subset).*

Hence for  $a > 0$ , the graph of  $t \mapsto \rho_a(t)$  is a continuous non decreasing Cantor like function whose graph is a so-called *devil's stair*. But, as proved locally by Arnold and globally by Herman, unless the original Cantor function, the subset where it strictly increases has positive Lebesgue measure. Let us call  $\bar{\rho}_a : [0, 1] \rightarrow [0, 1]$  the restriction of  $\rho_a$  to the interval  $[0, 1]$  :

**Proposition 35** ([A3, He2]) *For any  $a \neq 0$ , the Lebesgue measure of the Cantor set  $K_a = [0, 1] \setminus Int\bar{\rho}_a^{-1}(\mathbb{Q})$  is strictly positive and it tends to 1 when  $a$  tends to 0.*

The starting point is the following

**Exercise 15** *At a rotation  $R_\alpha$ , the map  $f \mapsto \rho(f)$  from  $D^0(\mathbb{T}^1)$  to  $\mathbb{R}$  is Lipschitz with Lipschitz constant 1.*

The bulk of the proof rests on Arnold's and Herman's smooth conjugacy theorems alluded to at the end of section 2.6.

### 3 Dynamics of area preserving monotone twists

#### 3.1 The big picture

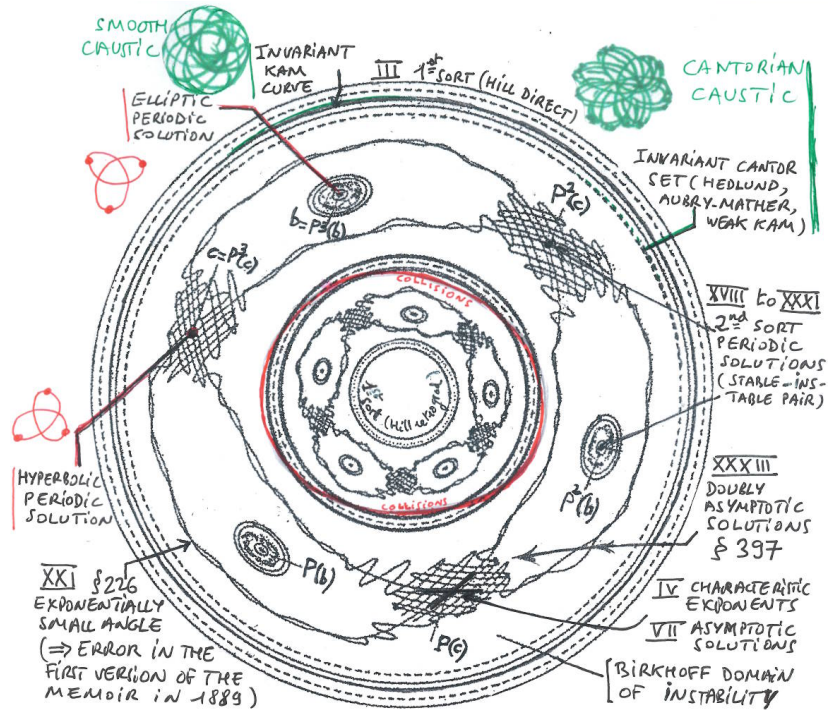


Figure 16. The return map of the restricted 3-body problem at high Jacobi constant (figure reproduced (slightly modified) with the kind permission of *Encyclopædia Universalis*).

The curves  $\Gamma_{\tau_0}$  given by theorem 6 form a Cantor family for which 0 is a density point (the relative measure of the Cantor set in smaller and smaller neighborhoods of 0 tends to 1). Nevertheless, this is far from being the whole story. The dynamics of such a generic area preserving  $F$  in the complement of the invariant curves (the so-called *Birkhoff domains of instability*) is extremely complicated and, if the works of Birkhoff, Aubry, Mather, Herman, have shed considerable light on the way invariant circles of the normal form break (periodic points, invariant Cantor sets, see [K, ?, LC2] and the references they contain), many questions remain.

Some of the complexity of a generic area preserving map of the disc is roughly suggested in figure 16, taken from [?]. It illustrates the dynamics of the monotone twist map of the annulus which arises when studying the *restricted three-body problem* at high values of the Jacobi constant (see section 6 of [C2] for explanations). To the periodic points are attached invariant stable (resp. unstable) manifolds along which the images of a point under the positive (resp. negative) iterates of  $F$  converge exponentially fast to the periodic orbit. The



homoclinic tangles (see [S] and section 10 of [?]) created by the intersections of such invariant manifolds produce invariant Cantor sets on which the dynamics of  $F$  is the same as the one of throwing a dice (more technically, a Bernoulli shift, see [LC1, C1, KH]) and hence possesses positive *topological entropy*. Also, orbits go from one boundary of a domain of instability to the other, but their *diffusion* is blocked by the invariant curves.

### 3.2 Definition and first examples

Let  $\bar{A}$  be the closed annulus  $\mathbb{T}^1 \times [0, 1]$ <sup>6</sup>. Let  $\bar{F} = (\bar{F}_1, \bar{F}_2) : \bar{A} \rightarrow \bar{A}$  be an orientation preserving homeomorphism. We call  $(x \in \mathbb{T}^1, y \in \mathbb{R})$  the natural coordinates in  $\bar{A}$ . We shall note  $F = (F_1, F_2)$  a lift of  $\bar{F}$  to the universal covering  $A = \mathbb{R} \times [0, 1]$ .

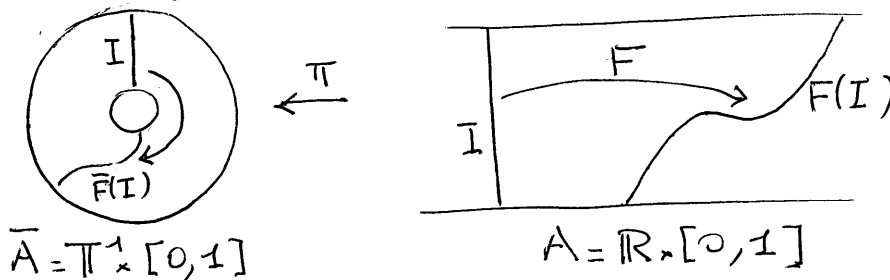


Figure 17.

**Definition 36** *The homeomorphism  $\bar{F}$  is said to be a positive monotone twist (in brief: a twist homeomorphism) if it preserves orientation and the boundary components and if for a lift  $F$  (and hence for all) the map  $y \mapsto F_1(x, y)$  from  $[0, 1]$  to  $\mathbb{R}$  is strictly increasing. If moreover  $F$  preserves the standard Lebesgue measure  $dx dy$  (or more generally a measure which weights positively any open set) one says it is conservative.*

In addition to the examples encountered in section 1 in the neighborhood of an elliptic fixed point, here are some examples.

(i) **Linear torsion.** A simple example is

$$F : A \rightarrow A, \quad F(x, y) = (x + y, y).$$

<sup>6</sup>The theory extends to the case where  $A$  is an open annulus  $\mathbb{T}^1 \times (0, 1)$  or  $\mathbb{T}^1 \times \mathbb{R}$  but we shall stick to the case of  $\mathbb{T}^1 \times [0, 1]$ . See also remark 3.4.4.

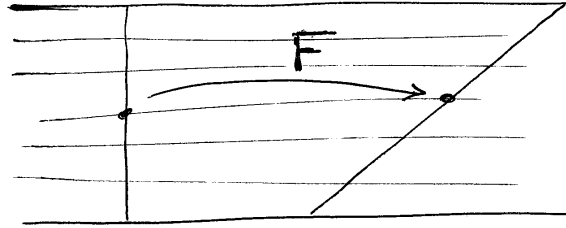


Figure 18.

(ii) **Pendulum map.** Another simple example is

$$F(x, y) = R_{p/q} \circ \varphi_t = \varphi_t \circ R_{p/q},$$

where  $R_{p/q}(x, y) = (x + p/q, y)$  and  $\varphi_t$  is the flow at a small positive time  $t$  of the pendulum-type differential equation

$$\ddot{x} + \omega^2 \sin(2\pi qx).$$

Figure 19, which illustrates the case  $p = 2, q = 3$ , features the level curves of the conserved “energy”  $H(x, y) = \frac{1}{2}y^2 - \frac{\omega^2}{2\pi q} \cos(2\pi qx)$ . To the singular points of  $H$  correspond two isolated untwined periodic orbits of period 3 and rotation number  $p/q = 2/3$ , one *hyperbolic*,  $\{z_0, F(z_0), F^2(z_0)\}$ , corresponding to an unstable equilibrium of the pendulum, and one *elliptic*,  $\{z_1, F(z_1), F^2(z_1)\}$ , corresponding to a stable equilibrium of the pendulum.

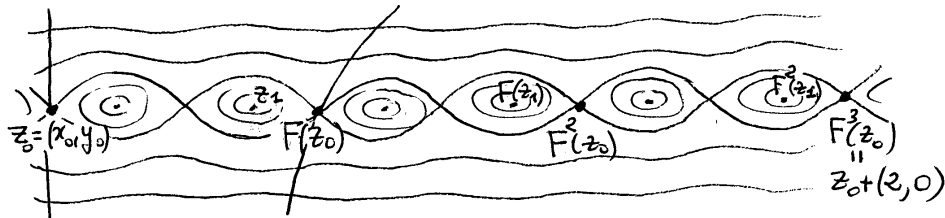


Figure 19. The pendulum map

(iii) **Poincaré return maps associated to a Hamiltonian system** As illustrated in section 3.1, the study of the dynamics of such conservative maps originates in the works of Henri Poincaré on the three-body problem.

There are many books introducing to Hamiltonian systems and their relation with mechanics. A nice start is Arnold’s classical [A4]. For a short introduction see [C2].

### 3.3 The billiard map

Given a convex billiard table, i.e. a compact convex domain of  $\mathbb{R}^2$  with smooth boundary  $\Gamma$ , a billiard trajectory is made of straight segments which reflect on

$\Gamma$  by changing the sign of the angle with the normal to  $\Gamma$  at the contact point. Such a trajectory is naturally associated to a map  $T : A \rightarrow A$  of the annulus  $A = S^1 \times [0, \pi]$  in the following way: let  $\gamma : [0, 2\pi] \rightarrow \mathbb{R}^2$  be a parametrization of  $\Gamma$  by arclength  $t$ . To a couple  $(t, \alpha)$  of a reflection point  $\gamma(t)$  and the reflection angle  $\alpha$ , the map  $T$  associates the couple  $(t_1, \alpha_1)$  corresponding to the next reflection.

**Exercise 16** Compute the billiard map  $T$  for a billiard in a round disc.

**Lemma 37** Let  $l(t, t_1)$  be the length of the Euclidean chord between the points  $\gamma(t)$  and  $\gamma(t_1)$ . One has

$$\frac{\partial l}{\partial t}(t, t_1) = -\cos \alpha, \quad \frac{\partial l}{\partial t_1}(t, t_1) = \cos \alpha_1$$

*Proof.* See figure 20 for a proof in the spirit of Newton.

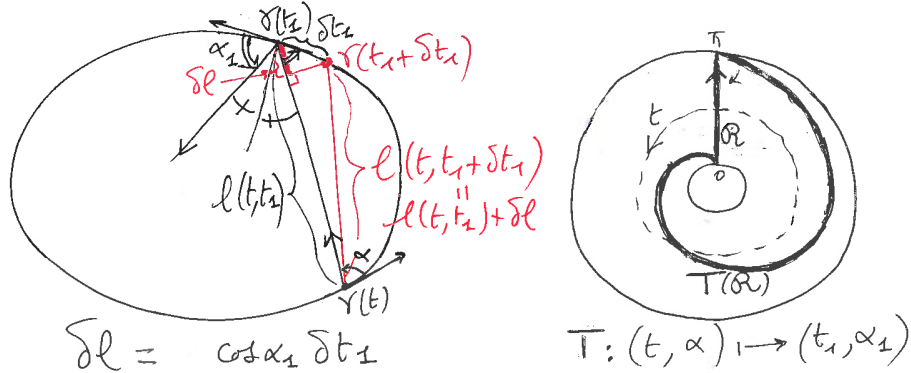


Figure 20. A convex billiard table and the associated billiard map.

**Corollary 38**  $T$  preserves orientation and the measure  $\sin \alpha \, d\alpha \wedge dt$ .

*Proof.*  $dl = -\cos \alpha \, dt + \cos \alpha_1 \, dt_1$ , hence  $0 = d^2l = \sin \alpha \, d\alpha \wedge dt - \sin \alpha_1 \, d\alpha_1 \wedge dt_1$ .

If one chooses the variables  $r = \cos \alpha$  and  $t$ , then the billiard map preserves orientation and the Lebesgue measure, that is the 2-form  $dr \wedge dt$ .

**Exercise 17** Using figure 20 show that  $\frac{\partial t_1}{\partial \alpha}(t, \alpha) = \frac{l(t, t_1)}{\sin \alpha_1}$ . Show that when  $\alpha$  tends to 0, this derivative tends to twice the radius of curvature of the billiard table at the point  $\gamma(t)$ .

We now show that the billiard map associated to an elliptic table is *integrable* in the sense that it admits a conserved quantity. For this, following [Ta], it is convenient to work with cartesian coordinates: the couple  $(t, \alpha)$  will be identified with a couple  $(x, v)$  where  $x = (x_1, x_2) \in \mathbb{R}^2$  describes the table's boundary and  $v = (v_1, v_2)$  is a unit vector directing the trajectory. Let

$$B(x, x) := \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$$

be the equation of the table's boundary

**Lemma 39** The quantity  $\frac{1}{2}B(x, v) = \frac{x_1 v_1}{a_1^2} + \frac{x_2 v_2}{a_2^2}$  stays constant along a trajectory of the elliptic billiard.

*Proof.* Writing  $T(x, v) = (x', v')$ , notice that  $B(x' + x, x' - x) = 0$ , hence  $B(x' + x, v) = 0$ . On the other hand,  $B(x', v' + v) = 0$  (see figure 21). Hence  $B(x', v') = -B(x', v) = B(x, v)$ .

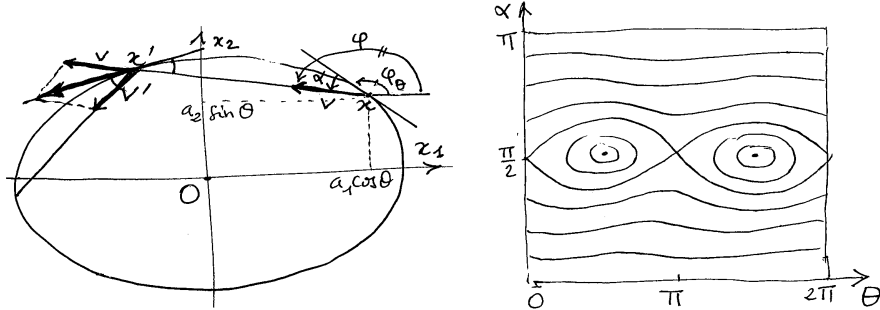


Figure 21. An elliptic billiard table and its conserved quantity (case  $a_1 > a_2$ ). The aim of the following exercise is to understand the nature of the simplest periodic orbits of the billiard map in an ellipse: the semi-major and semi-minor axes, which are the singularities of the conserved quantity defined in lemma 39. The level curves of this function are represented on figure 21 in terms of the parametrization of the domain of the billiard map by couples  $(\theta, \alpha) \in \mathbb{T}^1 \times [0, 1]$  defined as follows: we parametrize the boundary by the *eccentric anomaly*  $\theta$  and the set of unit velocity vectors by an angle  $\varphi$ , that is we set

$$x = (a_1 \cos \theta, a_2 \sin \theta), \quad v = (\cos \varphi, \sin \varphi).$$

The reflexion angle  $\alpha$  is then defined by the formula  $\alpha = \varphi - \varphi_\theta$ , where  $\varphi_\theta$  is the angle between the positive horizontal axis and the oriented tangent to the boundary at  $x$  (see figure 21).

**Exercise 18** 1) Show that when  $\theta$  tends to 0, one has

$$\varphi_\theta = \frac{\pi}{2} + \frac{a_1}{a_2} \theta + O_2(\theta).$$

2) Show that when  $\alpha$  tends to  $\frac{\pi}{2}$ , the leading non constant term of the conserved quantity is proportional to

$$(a_2^2 - a_1^2) \theta^2 + a_2^2 \left( \alpha - \frac{\pi}{2} \right)^2.$$

Deduce that the 2-periodic orbit defined by the semi-major (resp. semi-minor) axis is unstable (resp. stable) (compare to figure 21 and see [Ta] for a geometric interpretation).

**Remark.** The phenomenon which appears when deforming a circular billiard into an elliptic one is the *opening of a resonance zone*, already well understood by

Poincaré as the potential source of “non integrability” (compare to the comment on page 17): the invariant circle of the billiard map formed by the set of period two trajectories along any diameter of the circle breaks down into a pair of periodic trajectories along respectively the great and the small axis of the ellipse. Integrability is still not destroyed but yet this is the first step along the path which would, for a “generic” convex biliard table, transform the completely integrable twist map with all circles invariant into the complicated “big picture” of Figure 16.

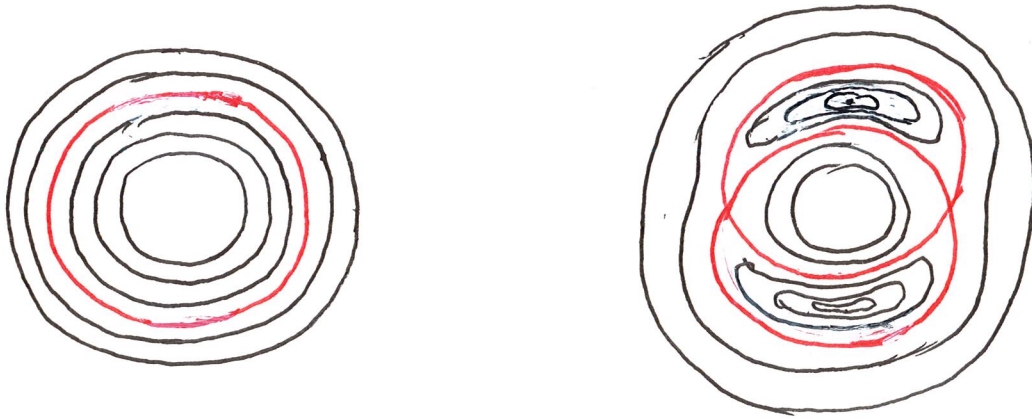


Figure 21-1. Opening of a resonance zone

The classical interpretation of the integrability of the elliptic billiard is the fact that any trajectory is tangent to a *caustic*, which is either a confocal ellipse or a confocal hyperbola, whose equation is of the form  $\frac{x_1^2}{a_1^2+\lambda} + \frac{x_2^2}{a_2^2+\lambda}$  for some  $\lambda$ . The elementary geometric proof ((see [Ta]) starts with the so-called focal property of the ellipse:

**Lemma 40** *If a segment of a billiard trajectory contains a focus, then every ssegment of this trajectory contains a focus.*

*Proof.* Because of the convexity of the ellipse, for any point  $P'$  on the tangent at  $P$  to the ellipse and any point  $Q$  on the ellipse, one has:

$$|P'F_1| + |P'F_2| > |QF_1| + |QF_2| = |PF_1| + |PF_2|.$$

This implies that, if  $F'_2$  is the symmetric of  $F_1$  with respect to the tangent at  $P$  to the ellipse, the points  $F_2, P, F'_2$  must be collinear (figure 21bis), which proves the lemma.

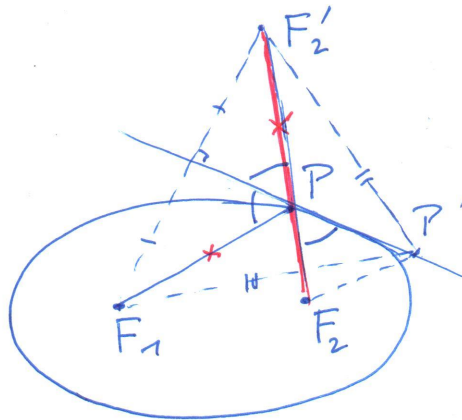


Figure 21-2. Focal property

**Exercise 19** Show that a billiard trajectory containing the foci must flatten in both directions and tend to the line joining the foci.

To which curves in figure 21 (right) such trajectories correspond ?

The end of the proof is illustrated on the following figure in the case of elliptic caustics; it consists in proving that two consecutive segments of the billiard trajectory are tangent to the same confocal ellipse because the two colored triangles are isometric, which implies  $|F_1'F_2| = |F_1F_2'|$ , that is  $c_1 = c_2$ :

$$c_1 = D_1 F_1 + D_1 F_2 = F_1' F_2$$

$$c_2 = D_2 F_1 + D_2 F_2 = F_2' F_1$$

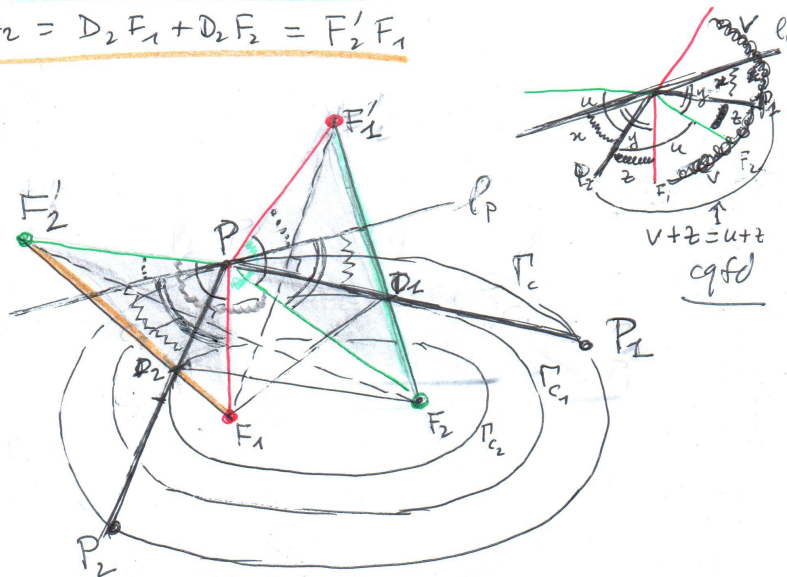


Figure 21-3. The proof



Figure 21-4 shows elliptic and hyperbolic caustics.

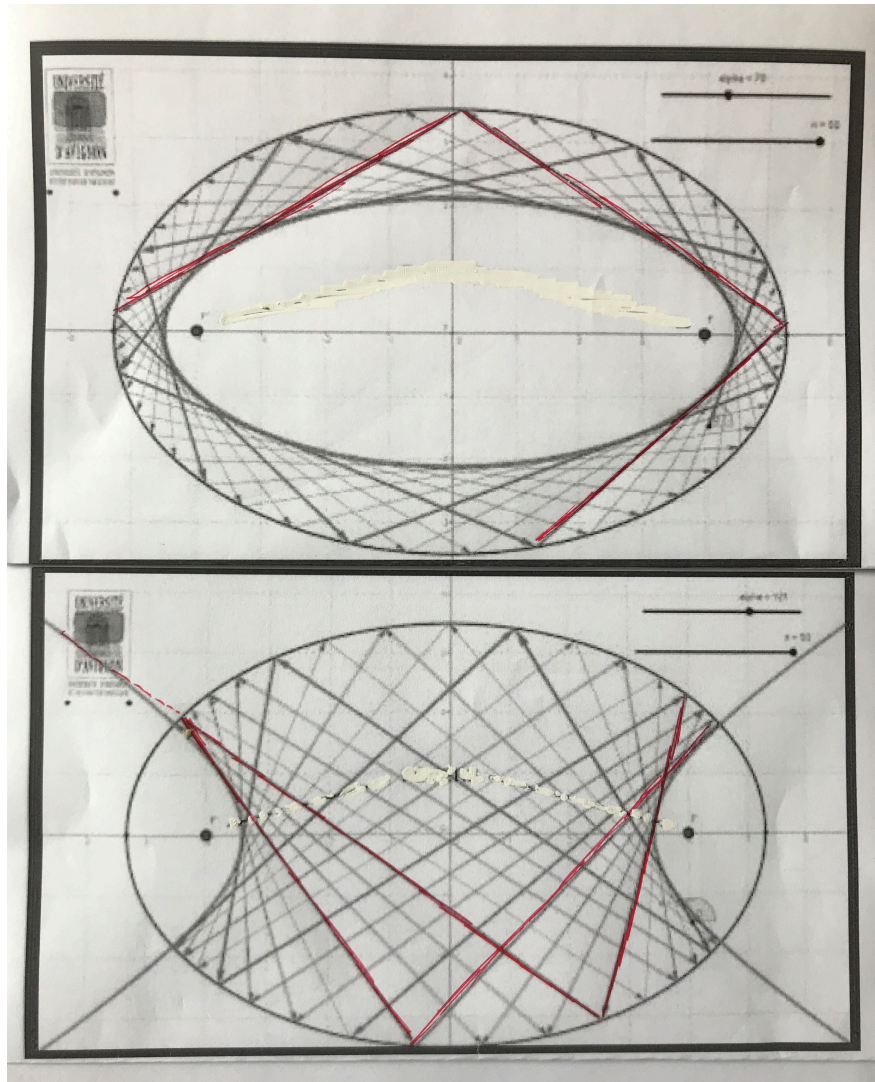


Figure 21-4. Caustics

Billiard trajectories crossing the segment  $[F_1F_2]$  are tangent to hyperbolas and the ones not crossing this segment are tangent to ellipses. The transition between the two kinds is made by the trajectories going through the foci, which generate the so-called *homoclinic orbits*.

**Exercise 20** To what curves in figure 21 (right) correspond the trajectories tangent to a confocal ellipse and the ones tangent to a confocal hyperbola ?

### 3.4 Aubry-Mather theory

In this section, we prove the existence of Birkhoff orbits and as a consequence the existence of Aubry-Mather invariant sets for any monotone area preserving twist map of the annulus. This generalizes Birkhoff results on the billiard map. An important generalization to higher dimensions exists : this is the so-called Weak KAM theory.

#### 3.4.1 Ordered invariant sets and Lipschitz estimates

In what follows, periodic orbits which are a natural generalization of the “hyperbolic” one in the pendulum map example (ii) will be obtained for a general conservative twist map of the annulus as minima of a certain functional. Generalizations of the “elliptic” one can also be obtained as minimax. Such ideas go back to Birkhoff’s works on billiards and were developed by Aubry and Le Daeron, Mather and Katok. We shall follow the simple proof given by Katok in [K], which works with a slightly more general definition of the word “conservative” ; indeed, it will be sufficient to suppose that  $F$  preserves a measure which is positive on open subsets.

The following definition, in which we follow [K], is directly inspired by the example of the pendulum map described in section 3.2(ii) (just label the hyperbolic – resp. elliptic – points in natural order):

**Definition 41** *Let  $p, q$  be relatively prime integers. A Birkhoff point of type  $(p, q)$  is a point  $z_0 = (x_0, y_0)$  in  $A$  whose orbit can be labeled in the following way: there is a sequence  $z_n = (x_n, y_n), n \in \mathbb{Z}$ , in  $A$ , whose projection  $x_n, n \in \mathbb{Z}$ , on  $\mathbb{R}$  is strictly monotone and which satisfies*

$$z_{n+p} = F(z_n), \quad z_{n+q} = z_n + (1, 0).$$

This implies that the projection  $\bar{z}_0$  of  $z_0$  on the annulus  $\bar{A}$  is a periodic point with rotation number  $p/q$ , that no two points of its orbits coincide and that they are ordered as the points in the orbit of the rotation  $(x, y) \mapsto (x + p/q, y)$ . Such an orbit is the simplest example of a  $F$ -ordered set as defined below:

**Definition 42** *If  $F = (F_1, F_2) : A \rightarrow A$  is the lift of a homeomorphism  $\bar{F}$  of the annulus  $\bar{A}$ , a subset  $M$  of  $A$  is said to be  $F$ -ordered if*

- 1)  $M$  is invariant under  $F$  and the integer translations  $T_{\pm 1}(x, y) = (x \pm 1, y)$ ;
- 2) the restriction to  $M$  of the projection  $\pi(x, y) = x$  is injective;
- 3) if  $(x, y)$  and  $(x', y')$  are two elements of  $M$  such that  $x < x'$ , one has  $F_1(x, y) < F_1(x', y')$ .

Being invariant under integer translations, an  $F$ -ordered set  $M$  projects to a  $\bar{F}$ -ordered invariant set  $\bar{M} \subset \bar{A}$ .

**Definition 43** *A minimal closed  $\bar{F}$ -ordered invariant set  $\bar{M} \subset \bar{A}$  will be called an Aubry-Mather set.*



**Exercise 21** Let  $\overline{M}$  be an Aubry-Mather set of  $\overline{F}$  and  $\overline{K} \subset \mathbb{T}^1$  be its projection on  $\mathbb{T}^1$ . Show that the restriction  $\overline{F}|_{\overline{M}}$  respects the cyclic order on  $\overline{M}$  and conclude that it is conjugate by a homeomorphism to the restriction to  $\overline{K}$  of a homeomorphism of  $\mathbb{T}^1$  and hence has a rotation number (in case of a Birkhoff orbit of type  $(p, q)$ , this rotation number is obviously equal to  $p/q \pmod{1}$ ). Deduce from section 2.4 that  $\overline{M}$  is

- either a Birkhoff periodic,
- or an invariant curve on which  $\overline{F}$  is conjugated to a rotation with irrational rotation number (which means dense orbits),
- or an invariant Cantor set.

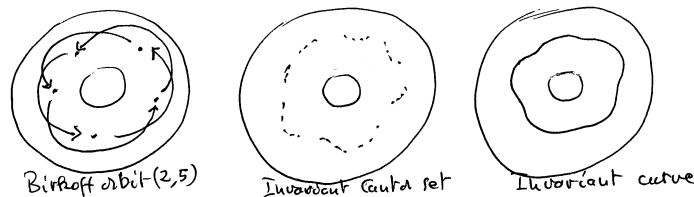


Figure 22. Aubry-Mather invariant sets

The fundamental property of  $F$ -ordered sets, whose origin goes back to Birkhoff's works on invariant curves, is stated in the following lemma:

**Lemma 44 (Lipschitz estimates)** Let  $F$  be the lift of a monotone twist which is Lipschitz. There exists  $l > 0$ , depending only on  $F$  such that, if  $M$  is  $F$ -ordered and if  $(x, y)$  and  $(x', y')$  belong to  $M$ , one has the uniform Lipschitz estimate

$$|y - y'| \leq l|x - x'|.$$

*Proof.* Let us suppose that  $y > y'$  (if not, replace  $F$  by  $F^{-1}$ ). The proof can be read on figure 23:

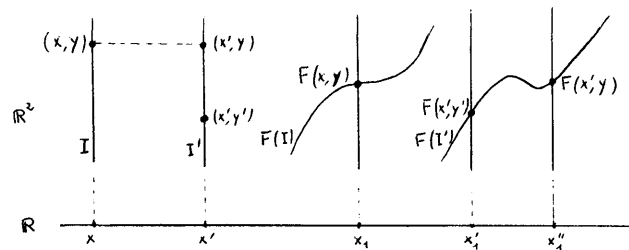


Figure 23.  $a(y - y') \leq x''_1 - x'_1 \leq x''_1 - x_1 \leq b(x' - x)$ .

The first inequality comes from the monotone twist condition, the second one from the fact that  $M$  is ordered, and the third one from the fact that  $F$  is supposed to be Lipschitz. In the perturbative case, when  $\overline{F}$  is close to an integrable map, one can get much better estimates for the Lipschitz constant (see [He3]).

**Exercise 22** Deduce from the Lipschitz estimates that the closure of a  $F$ -ordered subset is also  $F$ -ordered.

Interpolating linearly in the intervals of the complement of  $K$  in  $\mathbb{R}$  leads to the

**Corollary 45** *Supposing  $\overline{F}$  Lipschitz, any  $F$ -ordered subset  $M$  of  $A$  is contained in a Lipschitz graph (of  $\varphi : \mathbb{R} \rightarrow [0, 1]$ , resp.  $\overline{\varphi} : \mathbb{R} \rightarrow \mathbb{R}$ ) invariant under integer translations, hence projecting to a Lipschitz graph (of  $\varphi : \mathbb{T}^1 \rightarrow [0, 1]$ , resp.  $\overline{\varphi} : \mathbb{T}^1 \rightarrow \mathbb{R}$ ) in  $\overline{A}$  containing the projection  $\overline{M}$  of  $M$ .*

Soon after Aubry and Mather had proved the existence of such invariant sets for any rotation number, Katok made the fundamental remark that, because of the Lipschitz estimates, the existence of Mather sets of any irrational rotation number did follow from the existence of Birkhoff periodic orbits (see [K] and [KH] section 13.2). More precisely, recall the following

**Definition 46** *Let  $X$  be a compact metric space. The Hausdorff metric on the set of closed subsets of  $X$  is defined by the formula*

$$d(A, B) = \sup\{d(x, B), x \in A\} + \sup\{d(A, y), y \in B\}.$$

**Exercise 23** *Show that the Hausdorff metric defines a compact topology on the set of closed subsets of a compact metric space.*

**Exercise 24** *Show that the set of all Mather sets is closed in the Hausdorff topology and that the rotation number of a Mather set is continuous in this topology.*

*Hint 1:* use the fact that each Aubry-Mather set  $\overline{M}$  is contained in the graph of a Lipschitz function  $\Phi : \mathbb{T}^1 \rightarrow [0, 1]$  whose Lipschitz constant is bounded above by a quantity depending only on  $\overline{F}$  and recall the Arzela-Ascoli theorem.

*Hint 2:* use continuity of the rotation number of a homeomorphism of the circle under uniform limit.

From exercise 24 follows the

**Proposition 47** *Let  $\overline{F}$  be a monotone twist homeomorphism of the annulus which is Lipschitz and preserves a measure  $\overline{\mu}$  weighting positively open subsets. Then, in order that  $\overline{F}$  has an Aubry-Mather invariant set with rotation number  $\rho$ , it is sufficient that  $\overline{F}$  has Birkhoff periodic orbits of type  $(p_n, q_n)$  for a sequence  $p_n/q_n$  of rationals converging to  $\rho$ .*

**Remark.** In [K], Katok shows that even if  $F$  is not Lipschitz, Birkhoff periodic orbits and also Aubry-Mather sets are contained in graphs of continuous functions with a modulus of continuity which depends only on  $F$ . This property may be used in place of the Lipschitz estimates and hence the conclusion of the above Corollary holds without the hypothesis that  $F$  be Lipschitz.

### 3.4.2 Existence of Birkhoff periodic orbits: the variational principle

Let  $\overline{F} : \overline{A} \rightarrow \overline{A}$  be a conservative monotone twist map,  $F : A \rightarrow A$  is a lift to the covering space. The preservation of orientation and of the measure  $dxdy$

implies the preservation of the area 2-form  $dx \wedge dy$ . If  $F(x, y) = (x', y')$ , this can be written

$$dx' \wedge dy' = dx \wedge dy,$$

and implies (by the Poincaré lemma) the existence of a function  $h$  such that

$$dh(x, x') = -y(x, x')dx + y'(x, x')dx',$$

where  $y = y(x, x')$  and  $y' = y'(x, x')$  are uniquely defined by the condition that  $F(x, y) = (x', y')$ . On figure 24 is indicated a natural choice for  $h$  as the common area of the hatched triangles. Indeed, for any choice of  $h$  this area is both equal to  $h(x, x') - h(x, f_0(x))$  and  $h(x, x') - h(f_0^{-1}(x'), x')$ , which implies that it is equal to  $h(x, x') + C$  for some constant  $C$  (of course, it is also directly obvious that both functions  $h(x, f_0(x))$  and  $h(f_0^{-1}(x'), x')$  are constant).

Conversely,  $h$  defines  $F$  by

$$F\left(x, -\frac{\partial h}{\partial x}(x, x')\right) = \left(x', \frac{\partial h}{\partial x'}(x, x')\right).$$

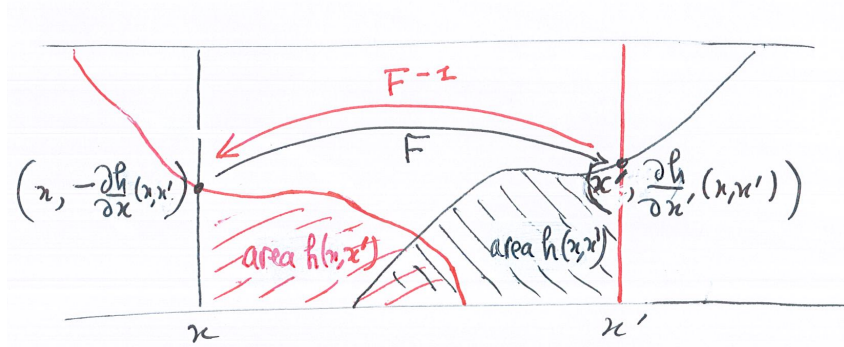


Figure 24.

Of course, if  $A$  is the closed annulus,  $h$  is defined only in the subset  $B$  of  $\mathbb{R}^2$  defined by

$$B = \{(x, x'), f_0(x) \leq x' \leq f_1(x)\},$$

where  $f_0$  and  $f_1$  are the restrictions of  $F$  to the boundaries  $\mathbb{R} \times \{0\}$  and  $\mathbb{R} \times \{1\}$  of  $A$ . Note that  $h$  is bounded below and such that  $h(x+1, x'+1) = h(x, x')$ . It is of class at least  $C^2$  and its hessian  $\frac{\partial^2}{\partial x \partial x'}$  is everywhere negative.

If  $p$  and  $q$  are integers, let  $X_{p,q}$  be the set of sequences

$$x = (x_i)_{i \in \mathbb{Z}} \quad \text{such that} \quad \forall i \in \mathbb{Z}, x_{i+q} = x_i + p.$$

The embedding  $X_{p,q} \rightarrow \mathbb{R}^q$  defined by  $x \mapsto (x_0, \dots, x_{q-1})$  induces a topology on  $X_{p,q}$ . Let  $W = W_{0,q} : X_{p,q} \rightarrow \mathbb{R}$  be defined by

$$W(x) = \sum_{i=0}^{q-1} h(x_i, x_{i+1}).$$

$W$  is invariant under integer translations, i.e.  $W(x) = W(T(x))$ , where

$$T(x) = (\bar{x}_i)_{i \in \mathbb{Z}} \quad \text{with} \quad \bar{x}_i = x_i + 1.$$

The quotient  $X_{p,q}/T$  is compact (under our hypotheses this is true for a finite annulus as well as for the infinite cylinder) and  $W$  is bounded below, hence it attains its minimum. If the minimum is in the interior of the domain  $B$ , it is a critical point, that is:  $\frac{\partial W}{\partial x_i} = 0$  for  $i = 0, 1, \dots, q-1$ . This implies that

$$\forall i \in \mathbb{Z}, \quad \frac{\partial h}{\partial x}(x_i, x_{i+1}) + \frac{\partial h}{\partial x'}(x_{i-1}, x_i) = 0,$$

and hence that  $(x_i, -\frac{\partial h}{\partial x}(x_i, x_{i+1}))$ ,  $i \in \mathbb{Z}$  is an orbit (see figure 25).

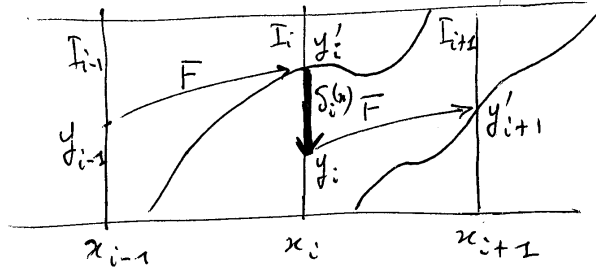


Figure 25.

In fact, an argument due to Aubry and Le Daeron shows that such an orbit is necessarily a Birkhoff orbit. All this works nicely in case  $\bar{A}$  is the infinite cylinder; in case  $A$  is a finite annulus, there are some technical problems due to the existence of a boundary for the domain of definition  $B$  of  $h$ .

We shall now restrict ourselves to the case of a closed annulus and explain the proof given by Katok, which solves in a very simple way – indeed without differential calculus – all these problems. As often in mathematics, it will be easier to solve a more general problem, namely the case when the preserved measure is just asked to weight positively each open subset and no regularity beyond continuity is required.

**Definition 48** The interval  $[\rho_0, \rho_1]$  defined by the rotation numbers  $\rho_i = \rho(F|_{\mathbb{R} \times \{i\}})$  of the restriction of  $F$  to the boundary of  $A$  is called the twist interval.

**Theorem 49** Let  $\bar{F}$  be a monotone twist homeomorphism of the annulus which preserves a measure  $\bar{\mu}$  weighting positively open subsets. Then  $\bar{F}$  has a Birkhoff periodic orbit of type  $(p, q)$  for any  $p/q$  belonging to the twist interval.

From Proposition 47 we deduce the

**Corollary 50** Let  $\bar{F}$  be a monotone twist homeomorphism of the annulus which is Lipschitz<sup>7</sup> and preserves a measure  $\bar{\mu}$  weighting positively open subsets. Then, for any  $\rho$  belonging to the twist interval,  $\bar{F}$  has an Aubry-Mather invariant set with rotation number  $\rho$ .

<sup>7</sup>The remark following Theorem 47 makes the hypothesis that  $\bar{F}$  be Lipschitz unnecessary.

*Proof.* Influenced by definition 41, we adapt the labeling of sequences to the expected behaviour of the orbit we are looking for: let  $M_{p,q}$  be the set of non decreasing bi-infinite sequences  $(x_n)_{n \in \mathbb{Z}}$  of real numbers such that, noting  $f_i = F|_{\mathbb{R} \times \{i\}}$ ,

$$x_{n+q} = x_n + 1 \quad \text{and} \quad f_0(x_n) \leq x_{n+p} \leq f_1(x_n).$$

The topology on  $M_{p,q}$  being induced by the embedding  $(x_n)_{n \in \mathbb{Z}} \rightarrow (x_0, \dots, x_{q-1})$ , its quotient  $M_{p,q}/T$  by the integer translations  $(x_n)_{n \in \mathbb{Z}} \mapsto (x_n + k)_{n \in \mathbb{Z}}$  is compact. That it is non empty can be seen in the following way: either  $p/q$  lies in the interior of the twist interval and  $\forall x, f_0^q(x) < x + p < f_1^q(x)$ , or it lies on the boundary  $\mathbb{R} \times \{i\}$  and  $\exists \tilde{x}, \tilde{x} + p = f_i^q(\tilde{x})$  (see Corollary 15). In the first case, one takes the sequence  $x_n$  obtained from the iterates of  $x$  under  $f_t$  for some homeomorphism  $f_t$  belonging to a monotone family interpolating between  $f_0$  and  $f_1$  and  $x$  arbitrary, while in the second case one takes the sequence obtained from the iterates of  $\tilde{x}$  under  $f_i$ .

Guided by the case when the lift  $\mu$  to  $A$  of the invariant measure is the Lebesgue measure  $dxdy$ , we define on  $M_{p,q}/T$  the functional

$$W((x_n)_{n \in \mathbb{Z}}) = \sum_{n=0}^{q-1} \mu(\tau(x_n, x_{n+p})),$$

where the ‘‘triangle’’  $\tau(x, x')$  is defined on figure 26:

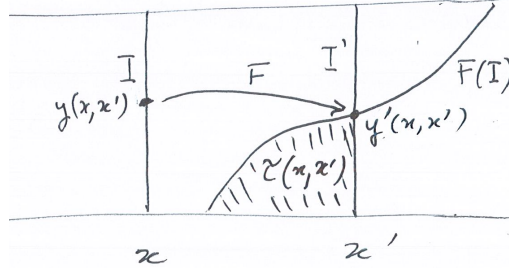


Figure 26.

The claim is that any local minimum of  $W$  is a Birkhoff orbit of type  $(p, q)$ . As such an orbit satisfies  $F(z_n) = z_{n+p}$ , it suffices to prove, as already explained (with different notations for the sequences), that at a local minimum of  $W$ , one has

$$\forall n \in \mathbb{Z}, \quad y(x_n, x_{n+p}) = y'(x_{n-p}, x_n),$$

where we recall that, if  $(x, x') \in B$ ,  $y(x, x')$  and  $y'(x, x')$  are uniquely defined by the equality  $F(x, y(x, x')) = (x', y'(x, x'))$ .

The proof of this equality is by showing that if it is not satisfied for some  $n$ , there exists a small perturbation of the sequence, which lowers  $W$ . The different cases are illustrated in figures 27 and 28.

We suppose first that  $x_{n-1} < x_n < x_{n+1}$  and  $y'(x_{n-p}, x_n) > y(x_n, x_{n+p})$  (left) or  $y'(x_{n-p}, x_n) < y(x_n, x_{n+p})$  (right). Moving a little  $x_n$  to the left or to the right we see that the preservation of  $\mu$  leads to a contradiction: indeed, in both cases, the sum  $\mu(\tau(x_{n-p}, x_n)) + \mu(\tau(x_n, x_{n+p}))$  has decreased; in the first case this is because the increase of  $\mu(\tau(x_n, x_{n+p}))$  is smaller than the decrease of  $\mu(\tau(x_{n-p}, x_n))$ , while in the second one, the decrease of  $\mu(\tau(x_n, x_{n+p}))$  is greater than the increase of  $\mu(\tau(x_{n-p}, x_n))$ .

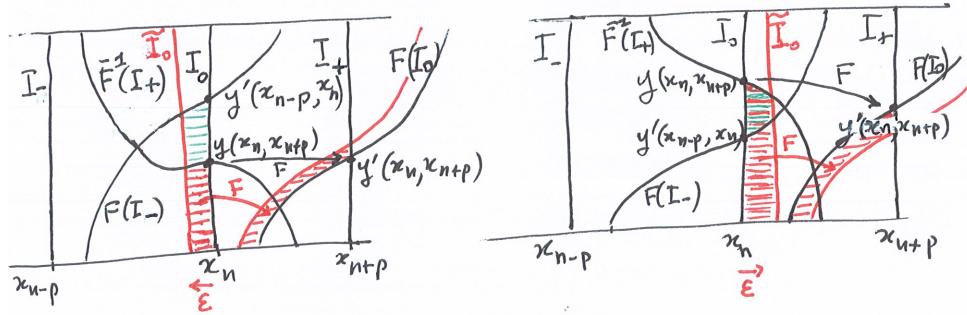


Figure 27.

Now suppose more generally that  $x_{n-1} < x_n = x_{n+1} = \dots = x_{n+k} < x_{n+k+1}$ . The twist property implies

$$1 \geq y'(x_{n-p}, x_n) \geq \dots \geq y'(x_{n-p+k}, x_{n+k}) \geq 0,$$

$$1 \geq y(x_{n+k}, x_{n+p+k}) \geq \dots \geq y(x_n, x_{n+p}) \geq 0,$$

hence either  $y'(x_{n-p}, x_n) \geq y(x_n, x_{n+p})$  or  $y(x_{n+k}, x_{n+p+k}) \geq y'(x_{n-p+k}, x_{n+k})$ , which is similar to the first case, or for all  $l$  between 0 and  $k$ ,  $y(x_{n+l}, x_{n+p+l}) = y'(x_{n-p+l}, x_{n+l})$ .

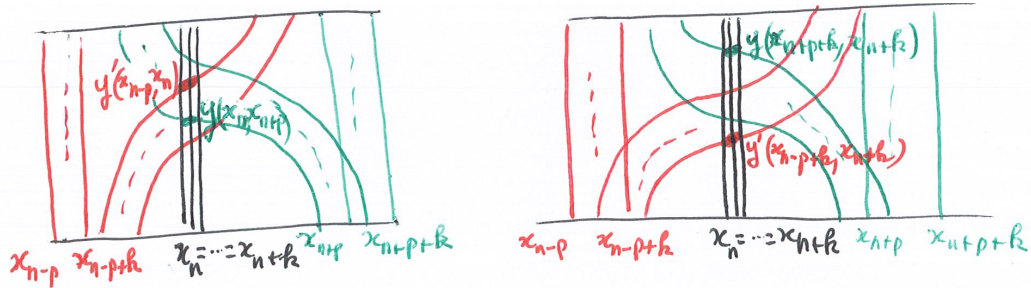


Figure 28.

Note that, in contrast with the use of differential calculus, the cases when some  $y(x_i, x_{i+p})$  or  $y'(x_{i-p}, x_i)$  belongs to the boundary have nothing special.

### 3.4.3 Homoclinic orbits

(This is a sketch, see [KH] section 13.2 for details). Now that we have obtained a Birkhoff periodic orbit of type  $(p, q)$  as a minimum of the action  $W$  on the

space  $M_{p,q}/T$ , let us come back to the example of the pendulum map (section 3.2 (ii)) and ask to which one of the two orbits represented on figure 19 it corresponds. For this, let us reverse the process used in the proof of existence of Aubry-Mather sets with irrational rotation number: to a sequence  $(\rho_n)_{n \in \mathbb{N}}$  of real numbers converging to  $p/q$  corresponds a sequence of Aubry-Mather sets  $(\overline{M}_n)_{n \in \mathbb{N}}$ ; Thanks to exercise 24, after possibly restricting to a subsequence one can suppose this sequence converges in the Hausdorff topology to an invariant closed subset  $\overline{M}$  of the annulus  $\overline{A}$ . If all the  $M_n$  are invariant circles, the limit is a circle on which  $\overline{F}$  is conjugate to the rotation  $R_{p/q}$ ; if not, it consists in orbits homoclinic to minimizing Birkhoff periodic orbit (or more generally heteroclinic to minimizing Birkhoff periodic orbits if there are several of them in the limit). This implies that the necessarily such limit periodic orbits are of hyperbolic type in the sense that they admit non periodic orbits of the same rotation number  $p/q$  asymptotic to them. But this does not imply that all minimizing Birkhoff periodic orbits are of this type (see [He3, L]).



Figure 29. Orbits homoclinic to a Birkhoff orbit.

#### 3.4.4 Remark: an important consequence of the Lipschitz estimates

Coming back to the initial example of twist maps arising from the study of conservative elliptic fixed points, a direct application of the above theory seems to require the existence of invariant closed annuli, that is the existence of invariant closed curves given by Moser's theorem 1.3.1. But, thanks to the Lipschitz estimates (lemma 44 and its corollary 45), this is not necessary: blowing up the fixed point via polar coordinates, one gets a conservative diffeomorphism  $\overline{F}$  of a closed annulus  $\overline{\mathbb{T}^1} \times [0, 1]$  onto its image as in figure 30 where only the lower boundary is preserved. Using generating functions it is possible to construct a conservative twist diffeomorphism  $\overline{G}$  of  $\overline{A}$  to itself which coincides with  $\overline{F}$  on a subannulus  $\overline{A}'$ . As the Lipschitz estimates give a good localization of the invariant sets of  $\overline{G}$  that we have constructed, we deduce that if their rotation numbers are close enough to the one of  $\overline{F}$  on the lower boundary, these invariant sets are located in  $\overline{A}'$  where  $\overline{G} = \overline{F}$ .

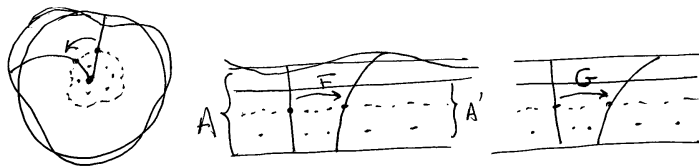


Figure 30.

In the same way, starting from a twist homeomorphism of the annulus, one can use the a priori localization of the Birkhoff orbits of type  $(p, q)$  to define a conservative twist homeomorphism of  $\mathbb{T}^1 \times \mathbb{R}$  with twist interval  $(\infty, +\infty)$  which

coincides with  $F$  on a subannulus  $\mathbb{T}^1 \times (\epsilon, 1 - \epsilon)$  which contains a priori any Birkhoff orbit (resp. Aubry-Mather set) of a given rotation number (see [KH] Proposition 9.3.5) and coincides for example with  $(x, y) \mapsto (x + y, y)$  outside  $\mathbb{T}^1 \times (0, 1)$ . This makes easier the proof in the next section.

### 3.4.5 The second type of Birkhoff orbit

(Again, this is a sketch, see [KH] section 9.3 for details). Inspired by the simple situation of the pendulum map (section 3.2 (ii)), we look for a second Birkhoff periodic orbit of type  $(p, q)$  which is intertwined with the one we just found in the sense that the projection on the circle of the two orbits would be the orbits of a homeomorphism of the circle with rotation number  $p/q$ . Technically we look for a minimax of the action in a well defined space; Birkhoff was the first to do that when studying periodic orbits of a billiard. In order to prove that any critical point of the functional  $W$  is a Birkhoff orbit, we shall suppose that  $F$  as well as the density of the invariant measure with respect to Lebesgue measure are of class  $C^1$  and that they have been extended to the cylinder as described in the former section. Then, the geometrical reasoning used for minima may be replaced by the fact that the functional  $W$  (introduced in the proof of Theorem 49) is differentiable and that any of its critical points corresponds to a Birkhoff orbit of  $F$ .

Let  $\Sigma = (x_n, y_n)_{n \in \mathbb{N}}$  be the lift of a Birkhoff orbit of type  $(p, q)$  obtained as a minimum of  $W$ . In order to constrain the orbit we are looking for, we define the space

$$M_{p,q}^\Sigma = \{(s_n)_{n \in \mathbb{N}}, s_{n+q} = s_n + 1 \text{ and } \forall n \in \mathbb{Z}, x_n \leq s_n \leq x_{n+1}\}.$$

This is a compact convex space on which the function  $W$  is defined. One proves first that the only critical points of  $W$  on the boundary are  $S = (x_n)_{n \in \mathbb{N}}$  and its translate  $S' = (x'_n)_{n \in \mathbb{N}}$  defined by  $x'_n = x_{n+1}$  and that the gradient vector-field of  $-W$  sends  $M_{p,q}^\Sigma$  into itself ([KH] Proposition 9.3.8). Elementary topology then allows to conclude that  $W$  possesses at least another critical point in the interior of  $M_{p,q}^\Sigma$ . Finally one shows that one of these new critical points must be a minimax ([KH] Proposition 9.3.9).

### 3.4.6 Readings

1) In the short paper [Ma], John Mather studies monotone twist mappings of the open cylinder  $\mathbb{T}^1 \times \mathbb{R}$  of the form

$$(x, y) \mapsto (x' = x + y + h(x), y' = y + h(x)), \quad \text{with } \int_0^1 h(x) dx = 0.$$

The condition on the integral of  $h$  is easily seen to be necessary for the existence of an invariant curve homotopic to the circles  $\mathbb{T}^1 \times a$ . By a very simple proof relying on Birkhoff's Lipschitz estimates for such an invariant curve, Mather shows that for Chirikov's *standard map*, which is the case when  $h(x) = \frac{k}{2\pi} \sin 2\pi x$ , no such invariant curve exists for  $k > 4/3$ .



Several authors studied criteria for the non existence of invariant curves with a given rotation number, see for example [Bo2]

2) In [Mo1], Jürgen Moser proves that any  $C^\infty$  twist map may be considered as the Poincaré return map of a time periodic Hamiltonian. The converse is not true, what plays the role of Hamiltonians satisfying the Legendre condition  $\frac{\partial L}{\partial q^2}$  being compositions of monotone twist maps.

## 4 In between circle and annulus homeomorphisms: degree one circle endomorphisms

When  $\bar{f}$  (hence any lift  $f$ ) is no more a homeomorphism, the limit  $\frac{1}{n}(f^n(x) - x)$  (see theorem 9) does not exist for every point  $x$  and only a rotation interval can be defined:

### 4.1 The rotation interval

Let  $\bar{f} : \mathbb{T}^1 \rightarrow \mathbb{T}^1$  be a continuous endomorphism of degree 1 of the circle and let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a lift of  $\bar{f}$ .

**Definition 51** *The rotation interval  $I(f)$  of  $f$  is defined as follows:*

$$I(f) = \left[ \inf_{x \in \mathbb{R}} \tilde{\rho}(x), \sup_{x \in \mathbb{R}} \tilde{\rho}(x) \right], \text{ where } \forall x \in \mathbb{R}, \quad \tilde{\rho}(x) = \limsup_{n \rightarrow \infty} \frac{1}{n} (f^n(x) - x).$$

*If in the definition of  $\tilde{\rho}(x)$  the limsup is in fact a limit, one calls  $\tilde{\rho}(x)$  the rotation number of  $x$  (or of its orbit) and one notes it  $\rho(x)$ .*

**Exercise 25** *Show that for a degree one endomorphism  $\bar{f}$  of the circle whose lifts are non decreasing, the rotation interval  $I(f) = \rho(f)$  reduces to a single point as in the case of a homeomorphism.*

The analogue of Birkhoff orbits (see definition 41), are naturally defined as follows:

**Definition 52** *As in the case of homeomorphisms of the annulus, we say that the orbit of  $x$  under  $f$  is  $\omega$ -ordered if for all  $m, m' \in \mathbb{N}$ ,  $n, n' \in \mathbb{Z}$ , the order of the points  $f^m(x) + n$  and  $f^{m'}(x) + n'$  on the line is the same as the one of  $m\omega + n$  and  $m'\omega + n'$ .*

One says also that the orbit under  $\bar{f}$  of  $\bar{x} \in \mathbb{T}^1$  on which  $x$  projects is  $\omega$ -ordered. Obviously, this property of  $\bar{x}$  is independent of the choice of the lift  $x$  of  $\bar{x}$ .

**Exercise 26** *Show that the orbits (resp. the periodic orbits if  $\rho(f)$  is rational) of a degree one endomorphism  $\bar{f}$  of  $\mathbb{T}^1$  whose lifts are non decreasing, are ordered on the circle as the orbits of the corresponding rotation.*

In fact,  $\omega$ -ordered orbits exist in general:

**Theorem 53** Given the lift  $f$  of a continuous endomorphism  $\bar{f}$  of degree 1 of the circle and its rotation interval  $I = [a, b]$ , for any  $\omega \in I(f)$ , there exists an orbit of  $f$  which is  $\omega$ -ordered.

The proof will use surgery: knowing the result of Exercise 26, one constructs the orbit we are looking for as an orbit of a non decreasing endomorphism which coincides with  $f$  along it and has the rotation number  $\omega$ .

**Proposition 54** ([CGT1]) Under the hypotheses of theorem 53, there exists a continuous family  $g_\mu$ ,  $\mu \in [0, 1]$ , of non decreasing endomorphisms of the circle with the following properties:

- 1)  $\forall \mu \in [0, 1]$ ,  $g_\mu$  is the lift of an endomorphism  $\bar{g}_\mu$  of the circle,  $g_0 \leq f \leq g_1$  and if  $g_\mu$  is not locally constant at  $x$ , one has  $g_\mu(x) = f(x)$ ;
- 2)  $I(g_0) = \{a\}$ ,  $I(g_1) = \{b\}$ .

*Proof.* [of proposition 54] We define  $g_0$  and  $g_1$  by the following formulas (see figure 31)

$$g_0(x) = \inf_{y \geq x} f(y), \quad g_1(x) = \sup_{y \leq x} f(y),$$

and then choose an interpolating family  $g_\mu$  as in figure 31 (the choice is not unique; formulas can be found in [CGT1]. This idea seems to have been used independently by several authors including P. Boyland [Bo1] who mentions R. Hall and L. Kadanoff).

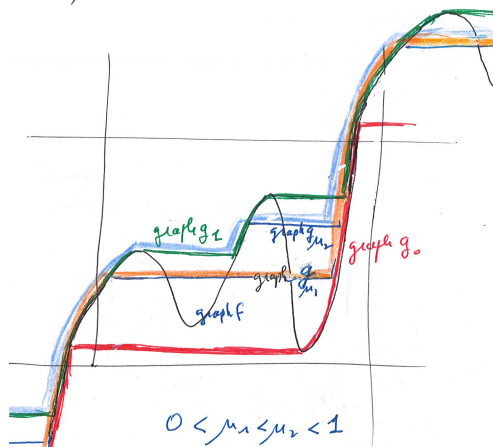


Figure 31. Defining a family  $g_\mu$

The  $g_\mu$  being non decreasing, each one has a well defined rotation number  $\rho_\mu$  (see exercise 25); moreover, the family  $g_\mu$  being continuous in the  $C^0$  topology, corollary 10 implies that these rotation numbers fill the interval  $[\rho_0, \rho_1]$  and, because  $g_0 \leq f \leq g_1$ , on has  $I(f) \subset [\rho_0, \rho_1]$ . The inclusion in the other direction will follow from theorem 53.

*Proof.* [of theorem 53] For any  $\omega \in I(f)$ , there exists  $\mu_\omega \in [0, 1]$  such that  $\rho_{\mu_\omega} = \omega$ . The orbits (only the periodic orbits if  $\omega \in \mathbb{Q}$ ) of  $\bar{g}_{\mu_\omega}$  being  $\rho_{\mu_\omega}$ -ordered (exercise 26), it is sufficient to show that, for each  $\mu \in [0, 1]$ , there exists  $\bar{y} \in \mathbb{T}^1$

whose orbit is contained in the subset of  $\mathbb{T}^1$  where  $\bar{g}_\mu$  coincides with  $\bar{f}$  (and is periodic if  $\rho(\mu) \in \mathbb{Q}$ ). Fixing  $\bar{x} \in \mathbb{T}^1$ , we call  $\{J_\alpha\}_{\alpha \in A}$  the set of open intervals in  $\mathbb{T}^1$  such that  $\bar{g}_{\mu_\omega}$  is locally constant (and hence differs from  $\bar{f}$ ) which are visited by the orbit of  $\bar{x}$ . We distinguish three cases:

- 1)  $\omega$  is irrational, and the set  $A$  is finite ;
- 2)  $\omega$  is irrational, and the set  $A$  is infinite ;
- 3)  $\omega = p/q$  is rational.

In the first case, there must exist  $n \in \mathbb{N}$  such that the orbit of  $\bar{y} = \bar{g}_{\mu_\omega}^n(\bar{x})$  does not meet any  $J_\alpha$ : if not, then for any  $n$  there must exist  $m \in \mathbb{N}$  such that  $\bar{g}_{\mu_\omega}^{mn}(\bar{x})$  belongs to some  $J_\alpha$ ,  $\alpha \in A$ . But this implies that there exists an infinite sequence  $p_1, p_2, \dots, p_k, \dots$  of integers such that  $\bar{g}_{\mu_\omega}^{p_k}(\bar{x})$  belongs to some  $J_\alpha$ . Hence, for at least two elements  $p_i, p_j$  of the sequence,  $\bar{g}_{\mu_\omega}^{p_i}(\bar{x})$  and  $\bar{g}_{\mu_\omega}^{p_j}(\bar{x})$  belong to the same  $J_\alpha$ , which implies that their image under  $\bar{g}_{\mu_\omega}$  is the same. Hence,  $\bar{g}_{\mu_\omega}^{p_i+1}(\bar{x}) = \bar{g}_{\mu_\omega}^{p_j+1}(\bar{x})$  which, if  $p_j > p_i$ , can be written  $\bar{g}_{\mu_\omega}^{p_j-p_i}(\bar{g}_{\mu_\omega}^{p_i+1}(\bar{x})) = \bar{g}_{\mu_\omega}^{p_i+1}(\bar{x})$ , that is  $\bar{g}_{\mu_\omega}^{p_i+1}(\bar{x})$  would be a periodic point, which contradicts the irrationality of  $\omega$ .

In the second case, let  $J_\alpha = ]\bar{a}_\alpha, \bar{b}_\alpha[$  and let  $\bar{y}$  be an accumulation point of the set  $\{\bar{b}_\alpha\}_{\alpha \in A}$ . The orbit of  $\bar{y}$  cannot enter into some open interval  $J$  where  $\bar{g}_{\mu_\omega}$  is constant: indeed, if  $\bar{g}_{\mu_\omega}^n(\bar{y}) \in J$ , the orbits of points belonging to some  $J_\alpha$  sufficiently close to  $\bar{y}$ , in particular points of the form  $\bar{g}_{\mu_\omega}^p(\bar{x})$  would also visit  $J$  which hence coincides with some  $J_{\alpha_0}$ ,  $\alpha_0 \in A$ . Hence the orbits of  $\bar{g}_{\mu_\omega}^n(\bar{y})$  and  $\bar{g}_{\mu_\omega}^p(\bar{x})$  coincide. But this would imply that eventually the orbit of  $\bar{y}$  should come back to some  $J_{\alpha_1}$  close enough to  $\bar{y}$  so that  $\bar{g}_{\mu_\omega}^n(J_{\alpha_1}) \subset J = J_{\alpha_0}$  and as in the first case, one concludes because there would exist an integer  $k$  such that  $\bar{g}_{\mu_\omega}(\bar{y}) = \bar{g}_{\mu_\omega}(\bar{g}_{\mu_\omega}^k(\bar{y})) = \bar{g}_{\mu_\omega}^k(\bar{g}_{\mu_\omega}(\bar{y}))$ .

In the third case, if a  $p/q$ -ordered (that is periodic) orbit  $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{q-1}$  of  $\bar{g}_{\mu_\omega}$  visits some open interval  $J$  where  $\bar{g}_{\mu_\omega}$  is constant, say  $\bar{x}_0 \in J$ , it consists in  $q$  fixed points of  $\bar{g}_{\mu_\omega}^q$  each of which belongs to some open interval where  $\bar{g}_{\mu_\omega}^q$  is constant. Indeed, if  $\bar{z}$  is close enough to  $\bar{x}_i$ ,  $\bar{g}_{\mu_\omega}^q(\bar{z}) = \bar{g}_{\mu_\omega}^i(\bar{g}_{\mu_\omega}^{q-i}(\bar{z}))$  and  $\bar{g}_{\mu_\omega}^{q-i}(\bar{z}) \in J$  has the same image as  $\bar{x}_0$  under  $\bar{g}_{\mu_\omega}$ , hence  $\bar{g}_{\mu_\omega}^q(\bar{z}) = \bar{g}_{\mu_\omega}^q(\bar{x}_i) = \bar{x}_i$ . One concludes by looking at figure 32: any intersection of the graph of  $\bar{g}_{\mu_\omega}^q$  with the diagonal outside of the intervals of constancy corresponds to a well ordered orbit of  $\bar{f}$ .

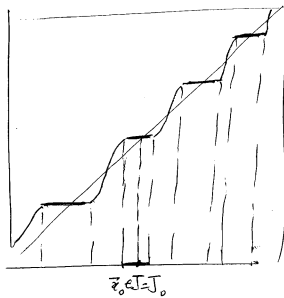


Figure 32. The rational case : graph of  $\bar{g}_{\mu_\omega}^q$ .

## 4.2 Ordered orbits in families

Let  $f_\mu$ ,  $\mu \in [0, 1]$ , be a continuous family of lifts of continuous endomorphisms  $\bar{f}_\mu$  of degree 1 of  $\mathbb{T}^1$  and let  $I(f_\mu) = [a_\mu, b_\mu]$  be the rotation interval of  $f_\mu$ .

**Theorem 55 ([CGT2])** *If  $b_0 < \omega < a_1$ , for any  $\bar{x}_0 \in \mathbb{T}^1$ , there exists  $\mu_0 \in ]0, 1[$  such that the orbit of  $\bar{x}_0$  under  $\bar{f}_{\mu_0}$  be  $\omega$ -ordered.*

**Exercise 27** *Show that in order to prove theorem 4.2, it is enough to consider the case when  $\omega = \frac{p}{q}$  is rational.*

*Proof.* The case  $q = 1$  being trivial (just look how the graphs of the fonctions  $f_\mu$  behave with respect to the graph of  $x \mapsto x + p$ ) we shall suppose that  $q \neq 1$ . We adopt the same ordering of candidate orbits as in the proof of corollary 50: supposing the fraction  $\frac{p}{q}$  irreducible (and different from an integer), let  $M_{p,q}$  be the set of non decreasing bi-infinite sequences  $(x_i)_{i \in \mathbb{Z}}$  of real numbers such that  $x_{i+q} = x_i + 1$ . With the topology induced by the embedding

$$(x_i)_{i \in \mathbb{Z}} \rightarrow (x_0, x_1 - x_0, x_2 - x_1, \dots, x_q - x_{q-1} = 1 + x_0 - x_{q-1}),$$

$M_{p,q}$  is homeomorphic to  $\mathbb{R} \times \Delta^{q-1}$ , where

$$\Delta^{q-1} = \left\{ y = (y_1, y_2, \dots, y_q, \forall i, y_i \geq 0, \sum_{i=1}^q y_i = 1) \right\}$$

is the standard  $(q-1)$ -dimensional simplex. Let  $M_{p,q}^{x_0}$  be the subspace of  $M_{p,q}$ , homeomorphic to the simplex  $\Delta^{q-1}$ , made of the elements with  $x_0$  given and let  $D : [0, 1] \times M_{p,q}^{x_0} \rightarrow \mathbb{R}^q$  be defined by

$$D(\mu, (x_i)_{i \in \mathbb{Z}}) = (d^1, d^2, \dots, d^q), \quad d^i = x_{i+p} - f_\mu(x_i).$$

**Exercise 28** *If  $q \neq 1$ , the image under  $D$  of the boundary  $\partial([0, 1] \times M_{p,q}^{x_0}) = \{0, 1\} \times M_{p,q}^{x_0} \cup [0, 1] \times \partial M_{p,q}^{x_0}$  of  $[0, 1] \times M_{p,q}^{x_0}$  does not contain  $0 \in \mathbb{R}^q$ .*

*Indication.*  $\frac{p}{q}$  being irreducible, there exists  $i_0, j_0 \in \mathbb{Z}$  such that  $i_0 p + j_0 q = 1$ . Hence  $D((x_i)_{i \in \mathbb{Z}}) = 0$  implies that, for all  $i \in \mathbb{Z}$ ,  $x_{i+1} = f^{i_0}(x_i) + j_0$ . Conclude that, if  $x_i = x_{i+1}$ , the point  $\bar{x}_i \in \mathbb{T}^1$  is fixed by  $\bar{f}$  and get a contradiction.

**Exercise 29** *Show that one can choose a fonction  $\rho : [0, 1]^2 \rightarrow \mathbb{R}$  such that the 2-parameter family  $f_{\mu,t}$  of lifts of continuous endomorphisms of degree 1 of  $\mathbb{T}^1$  defined by  $f_{\mu,t} = Id + [\sup(0, 1 - 2t) \times (-Id + f_\mu)] + \rho_{\mu,t}$  be such that*

- 1)  $f_{\mu,0} = f_\mu$ ,
- 2) For each  $t \in [0, 1]$ ,  $f_{\mu,t}$  satisfies the hypotheses of theorem 4.2,
- 3)  $f_{\mu,1}$  is a monotonous family of rotations.

*End of the proof of theorem 4.2:* Exercise 29 suggests that we first study the case of a monotonous family of rotations  $f_\mu(x) = x + \mu$  where  $\mu$  belongs to an interval  $[\mu_0, \mu_1]$  containing  $\frac{p}{q}$ . Notice that in this case,  $D$  is the affine diffeomorphism

$$D(\mu, (x_i)_{i \in \mathbb{Z}}) = (x_{1+p} - x_1 - \mu, x_{2+p} - x_2 - \mu, \dots, x_{q+p} - x_q - \mu) \text{ and that } D\left(\frac{p}{q}, (x_0 + \frac{i}{q})_{i \in \mathbb{Z}}\right) = (0, 0, \dots, 0).$$

**Exercise 30** Conclude the proof of theorem 4.2 in the case  $q \neq 1$  by using the homotopy  $t \mapsto f_{\mu,t}$  to show that, for the original family  $f_\mu$ , the degree of the map  $D$  from the boundary of  $[0, 1] \times M_{p,q}^{x_0}$  to  $\mathbb{R}^q \setminus 0$  (exercise 28) is  $\pm 1$ .

**Reading: the Arnold family beyond the domain of homeomorphisms.**

The continuation of Arnold's tongues in the part of Arnold's family (section 2.8) consisting of non injective endomorphisms is thoroughly studied in [Bo1]. It is proved there that the diagram on figure 15 extends nicely above the line  $a = \frac{1}{2\pi}$ : the tongues overlap in a uniform, monotonic manner and, for irrational values of  $\rho_a$ , they open into a tongue whose tip lies on the line  $a = \frac{1}{2\pi}$  (figure 33).

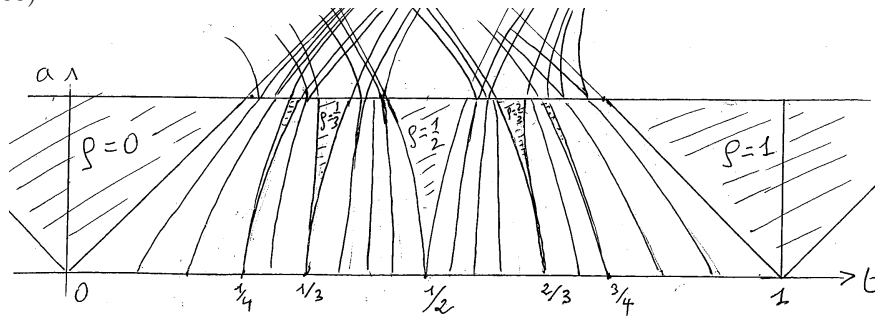


Figure 33. Tongues overlap in the Arnold family.

### 4.3 Rotation interval and the rotation sets of individual orbits

Given a continuous degree one endomorphism  $\bar{f}$  of the circle, and a lift  $f$  of  $\bar{f}$ , one can define the *rotation set*  $\rho(f, x)$  of an individual point  $\bar{x} \in \mathbb{T}^1$  as the set of limit points of  $\frac{1}{n}(f^n(x) - x)$  where  $x$  is any lift to  $\mathbb{R}$  of  $\bar{x}$ .

**Theorem 56 ([BMPT])** 1) For any  $\bar{x} \in \mathbb{T}^1$ ,  $\rho(f, \bar{x})$  is a closed subinterval of  $\rho(f)$ .

2) Given  $[\alpha, \beta] \subset \rho(f)$ , there exists  $\bar{x} \in \mathbb{T}^1$  such that  $\rho(f, \bar{x}) = [\alpha, \beta]$ .

A simple example is given by  $f(x) = x + \sin 2\pi x$ , which belongs to the Arnold's family (see section 2.8):

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